Mr's G's Little Book on

Tuning Systems

or why everything is out of tune

Summary

History from Pythagoras to modern day has seen a variety of tuning systems come and go. Just as the heart of mathematics has a fatal "hole" – no system can be consistent and complete at the same time – all musical instruments are cursed to be out of tune with someone.

The Four Primary Systems

Pythagorean Tuning – developed by Pythagoras around 5^{th} century BC

Ptolemaic Tuning – developed by Ptolemy around the 1^{st} century AD.

Mean Tone Tuning sneaked in by organ tuners around the 17^{th} century much to the chagrin of others – seeing it creating as many new problems as it solved. The mathematics of this came after the practice.

Equal Tempered Tuning where everyone gives up the argument and settles for everything being out of tune but not enough for the masses to notice – especially if played fast.

Worship Group Tuning

In a nutshell, it is likely that the organ, the grand piano and the guitar are all tuned to different systems. Only the violin is free to pick and choose.

Organ tuners like long slow deep notes so seek to avoid those horrible "wolf" beat frequencies. They squeeze down the fifths just a little so the beats are too long to be noticed but keep the octaves in tune.

The grand piano tuner gives that instrument the highest status. It needs no accompaniment anyway. They sneak up the top end retaining the perfect 5^{th}s but abandoning the octaves. No pianist hits top C and bottom C at the same time and when playing an arpeggio through the entire range the congregation will have forgotten the first note by the time the last is played.

The guitarist is stuck with an imperfect instrument where all notes other than the open strings are out of tune. They probably never realise anyway being too engrossed with their latest effects pedal.

At best the instrumentalist who has power of exact tuning – the violinist and the skilled wind instrumentalist – either consciously or unconsciously switches between the systems depending with whom they are playing or are just grateful to be close enough so that everyone's happy.

Pythagorean Tuning

Pythagoras takes the octave as the purest interval at 2:1 and the 5th as the next at 3:2. As a 5th plus a 4th must make the octave – termed a complementary pair - the 4th is set at 4:3.

The major 2nd 3rd 6th and 7th intervals are derived from multiples of 5ths while the diminished 2nd 3rd 5th 6th and 7th intervals are derived from multiples of 4ths. Thus one octave, one fundamental, one consequential and nine derived gives the twelve intervals between the thirteen notes from say C to C'.

The properties of the Pythagorean scale are

1) The complementary pairs giving the octaves are all perfect with the exception of two diminished 5ths which for any whole ratio system will always fail.

2) The "scale" whole tone intervals are equal at 204 cents ($^{9}/_{8}$). Only E to F$^{\#}$/G^{b} is destined to fail at 180 cents ($^{65536}/_{59049}$).

3) The two scale half tone intervals are equal at 90 cents ($^{256}/_{243}$) together with five others. The five remaining calculate at 114 cents ($^{2187}/_{2148}$)

4) The major 3rd is exactly two whole tones

5) Two maj.5ths are exactly equal to an octave plus a whole tone and any multiple thereof.

6) Four maj.5ths are exactly equal to two octaves plus a maj.3rd.

However two half tones do not and cannot equal a tone because the half tone is derived from 4ths and the tone from 5ths

Two dim.5ths falling short of an octave by 24 cents is a consequence of the irrationality of $\sqrt{2}$. No tuning system stemming from whole numbers will achieve this.

Pythagorean tuning also fails in that no combination of only 4ths or only 5ths can ever perfectly combine to give whole octaves. They contain only powers of 2 whereas maj.4ths and maj.5ths contain powers of 3. Specifically twelve maj.5ths exceed seven octaves by roughly 23½ cents and twelve maj.4ths fall short of five octaves by the same amount. This is termed the Pythagorean comma of $^{531441}/_{524288}$ which also is the error between 6 whole tones or three maj.3rds and an octave.

So right at the start of our investigation we can foresee that any tuning system based around whole numbers is destined to fail in that some combined intervals will lack perfect intonation.

Ptolemaic Tuning

Ptolemy recognised that Pythagoras's blind obedience to "small" integers did not allow sufficient fine tuning. So he set out to define the intervals by encompassing the number 5. He then determined the nearest available whole number ratio for each interval.

The starting point is to define the major 3rd as $^5/_4$ rather than the cumbersome $^{81}/_{64}$ as well as retaining the major 5th at $^3/_2$. The outcome is that we get a better scale for solo singing but hopelessly out for the "barber shop" quartets.

On the plus side, apart from the intervals sounding more "pure", two octaves plus a maj.3rd give exactly the ratio 5:1. Of the complementary intervals only the maj.2nd plus the dim.7th now fails. It is often considered the only tuning that can be reasonably sung.

Mean Tone Tuning

Two octaves plus a 3^{rd} is $2^2 \times {}^5/_4 = 5$ which should really be four 5^{th}s if God had taken a little more care in His design of the Universe. To correct this oversight organ tuners set the 5^{th} to be $\sqrt[4]{5} = 1.495$. They didn't consciously do the maths at the outset – just the practicalities. This is fractionally shy of the Pythagorean and Ptolemaic systems of ${}^3/_2$.

If we now wish to recover the principles of the Pythagorean tuning where the scale tones and half tones are equally spaced we just set the 4^{th} to $2 / \sqrt[4]{5}$ or by rationalising the denominator to $2.\sqrt{5}.\sqrt[4]{5} / 5$. The definitions of all the Pythagorean intervals can now be repeated to establish equally spaced intervals. The tone becomes $\frac{1}{2}\sqrt{5}$ or 193 cents and the half tone interval at $8.\sqrt{5}.\sqrt[4]{5} / 25$ or 117 cents. Again two dim.5^{th}s do not give an octave but we recover the maj.2^{nd} plus dim.7^{th} as a complementary pair. The chief disadvantage of mean tone tuning is that the scale half tones are pushed from 90 cents in the Pythagorean scale to 117 cents in the mean tone scale plus five other half tone intervals. The remaining five half tone intervals drop to 76 cents.

This has consequences. The $G^{\#}$ to E^{b} maj.5^{th} in the key of C is now nearly two commas out of tune or half a semitone. Even the most cloth-eared can hear that. The "resultant "beating" gave it the name the "wolf" interval. Some organs had two keys – one for $G^{\#}$ and another for A^{b} just to get round this problem.

On balance mean tone tuning might be heading in the right direction but starting off on the wrong path.

Equal Temperament

Whether by accident or design the mean tone system slowly gave way to equal temperament to eliminate the occurrence of wolf intervals. Instead of heaping the sins on one note, equal temperament abandons whole note ratios and sets every key equally out of tune. Specifically equally spacing all twelve half notes sets the ratio at $^{1}/_{12}\sqrt{2} \cong 1.0595$. This is an irrational number so can never be recreated from whole number ratios.

The gain is that all "mistakes" are spread throughout the octave. Two dim.5ths now equate to an octave as do all the complementary pairs as expected. The principle of equal tones and half tones of the Mean Tone system are retained. What is lost is the Ptolemaic octave plus a maj.5th no longer gives 1:3 and two octaves plus maj.3rd 1:5.
If you tune the bass strings of a guitar "exactly" and then bring out the harmonic on the 5th fret of one string to the 7th of the next you can hear the beats indicating something is still out of tune.

Conclusion

No tuning system solves all the problems. Each is a compromise depending on the priorities – solo playing or orchestra, solo singing or "barber shop" quartets.

Theological Consequences

So which system is used in Heaven or the New Earth? Preachers have told me not only was Satan the highest ranking created being, he may also have been the first worship leader. I have heard suggestions his actual body was designed to be a musical instrument though whether tuned to the Pythagorean, Ptolemaic, Mean Tone or Equal Temperament system has never been made clear. His fall is also a salutary warning to all worship group leaders.

When I'm eventually worshipping in Heaven – particularly if I get assigned to the worship group - do I need to worry about tuning? It would be hard to imagine God would be bound by an imperfect tuning system. However it would be even harder to conceive a place where the basic rules of arithmetic do not hold. Does God break the rules or create new ones that allow perfect tuning?

It would be great to have a conversation with Laplace about it though I doubt he'll have made it - not having any need for that hypothesis.

In a dream I once saw myself with a group of players. The pianist was at a keyboard nearly one 100 feet long. Miraculously the top end was still in tune with the bottom - not only in all the octaves but all the 5^{th}s as well.

But what really caught my ear was the drummer. He was just phenomenal.

I whispered to a nearby angel – "That must surely be Buddy Miles?".

"No" he replied. "That's God. He just thinks he's Buddy Miles."

Aspect	**Pythag.**		**Ptol.**		**Mean Tone**		**Temp.**	
Definition Check								
dim.2nd = 5×4th –2×octave	$^{256}/_{243}$	✓	$^{16}/_{15}$	×	$^{8.\frac{3}{4}\sqrt{5}}/_{25}$	✓	$2^{\wedge 1/12}$	✓
maj.2nd = 2×5th –1×octave	$^{9}/_{8}$	✓	$^{9}/_{8}$	✓	$\frac{1}{2}\sqrt{5}$	✓	$2^{\wedge 1/6}$	✓
dim.3rd = 3×4th –1×octave	$^{32}/_{27}$	✓	$^{32}/_{27}$	×	$^{4.\frac{1}{4}\sqrt{5}}/_{5}$	✓	$2^{\wedge 1/4}$	✓
maj.3rd = 4×5th –2×octave	$^{81}/_{64}$	✓	$^{81}/_{64}$	×	$^{5}/_{4}$	✓	$2^{\wedge 1/3}$	✓
maj.4th = 1×4th	$^{4}/_{3}$	✓	$^{4}/_{3}$	×	$^{2.\frac{3}{4}\sqrt{5}}/_{5}$	✓	$2^{\wedge 5/12}$	✓
dim.5th = 6×4th –2×octave	$^{1024}/_{729}$	✓	$^{1024}/_{729}$	×	$^{16.\sqrt{5}}/_{25}$	✓	$2^{\wedge 1/2}$	✓
maj.5th = 1×5th	$^{3}/_{2}$	✓	$^{3}/_{2}$	×	$^{\frac{1}{4}}\sqrt{5}$	✓	$2^{\wedge 7/12}$	✓
dim.6th = 4×4th –1×octave	$^{128}/_{81}$	✓	$^{128}/_{81}$	×	$^{8}/_{5}$	✓	$2^{\wedge 2/3}$	✓
maj.6th = 3×5th –1×octave	$^{27}/_{16}$	✓	$^{27}/_{16}$	×	$\frac{1}{2}{}^{\frac{3}{4}}\sqrt{5}$	✓	$2^{\wedge 3/4}$	✓
dim.7th = 2×4th	$^{16}/_{9}$	✓	$^{16}/_{9}$	×	$^{4.\sqrt{5}}/_{5}$	✓	$2^{\wedge 5/6}$	✓
maj.7th = 5×5th –2×octave	$^{243}/_{128}$	✓	$^{81}/_{64}$	×	$^{5.\ \frac{1}{4}\sqrt{5}}/_{4}$	✓	$2^{\wedge 11/12}$	✓
octave	$^{2}/_{1}$	✓	$^{2}/_{1}$	✓	$^{2}/_{1}$	✓	$^{2}/_{1}$	✓
whole tone	$^{9}/_{8}$		*varies*		$\frac{1}{2}\sqrt{5}$		$2^{\wedge 1/6}$	
half tone	$^{256}/_{243}$		$^{16}/_{15}$		$^{8.\frac{3}{4}\sqrt{5}}/_{25}$		$2^{\wedge 1/12}$	
Complementary Check								
dim.2nd + maj.7th		✓		✓		✓		✓
maj.2nd + dim.7th		✓		×		✓		✓
dim.3rd + maj.6th		✓		✓		✓		✓
maj.3rd + dim.6th		✓		✓		✓		✓
maj.4th + maj.5th		✓		✓		✓		✓
dim 5th + dim.5th		×		×		×		✓

Whole Ratio Intervals

1×octave + maj.5th = 1:3	✓	✓	×	×
2×octave + maj.4th = 1:5	×	✓	×	×

Other Paired Check

1× whole = two halves	×	×	×	✓
1× maj.3rd = 2 whole tones	✓	×	✓	✓
2× maj.5th = 1 octave plus tone	✓	×	✓	✓
4× maj.5ths = 2 octaves plus maj.3rd	✓	×	✓	✓
12× maj 5ths = 7 octaves	×	×	×	✓
10× maj 4ths = 5 octaves	×	×	×	✓