

Fibonacci Properties – Paper I

Summary

It is an interesting property that for any Fibonacci number F_k then

$$F_{nk} \bmod F_k = 0$$

that is every multiple of k in F_k is divisible by F_k . Here is a summary to F_{20} .

$$F_0 = 0$$

$$F_1 = 1 \text{ and obviously applies for } F_1.$$

$$F_2 = 1 \quad \text{index prime so n/a}$$

$$F_3 = 2 \quad \text{index prime so n/a}$$

$$F_4 = 3 \quad 3 \times F_2$$

$$F_5 = 5 \quad \text{index prime so n/a}$$

$$F_6 = 8 \quad 4 \times F_3$$

$$F_7 = 13 \quad \text{index prime so n/a}$$

$$F_8 = 21 \quad 7 \times F_4$$

$$F_9 = 34 \quad 17 \times F_3$$

$$F_{10} = 55 \quad 11 \times F_5$$

$$F_{11} = 89 \quad \text{index prime so n/a}$$

$$F_{12} = 144 \quad 72 \times F_3 \quad 18 \times F_6$$

$$F_{13} = 233 \quad \text{index prime so n/a}$$

$$F_{14} = 377 \quad 29 \times F_7$$

$$F_{15} = 610 \quad 305 \times F_3 \quad 122 \times F_5$$

$$F_{16} = 1597 \quad 47 \times F_8$$

$$F_{17} = 1597 \quad \text{index prime so n/a}$$

$$F_{18} = 2584 \quad 1292 \times F_3 \quad 323 \times F_6 \\ 76 \times F_9$$

$$F_{19} = 4181 \quad \text{index prime so n/a}$$

$$F_{20} = 6765 \quad 1353 \times F_5 \quad 123 \times F_{10}$$

Initial Investigations

$$F_{2n+2} = F_{2n+1} + F_{2n+0}$$

$$F_{3n+3} = F_{3n+2} + F_{3n+1} \\ = F_{3n+1} + F_{3n+0} + F_{3n+1}$$

$$F_{3n+3} = 2 F_{3n+1} + 1 F_{3n+0}$$

$$F_{4n+4} = F_{4n+3} + F_{4n+2} \\ = F_{4n+2} + F_{4n+1} + F_{4n+1} + F_{4n+0} \\ = F_{4n+1} + F_{4n+0} + 2 F_{4n+1} + F_{4n+0}$$

$$F_{4n+4} = 3 F_{4n+1} + 2 F_{4n+0}$$

$$F_{5n+5} = F_{5n+4} + F_{5n+3} \\ = F_{5n+3} + F_{5n+2} + F_{5n+2} + F_{5n+1} \\ = F_{5n+3} + 2 F_{5n+2} + F_{5n+1} \\ = F_{5n+2} + F_{5n+1} + 2 F_{5n+1} + 2 F_{5n+0} \\ \quad \quad \quad + F_{5n+1} \\ = F_{5n+2} + 4 F_{5n+1} + 2 F_{5n+0} \\ = F_{5n+1} + F_{5n+0} + 4 F_{5n+1} + 2 F_{5n+0}$$

$$F_{5n+5} = 5 F_{5n+1} + 3 F_{5n+0}$$

$$F_{6n+6} = F_{6n+5} + F_{6n+4} \\ = F_{6n+4} + F_{6n+3} + F_{6n+3} + F_{6n+2} \\ = F_{6n+4} + 2 F_{6n+3} + F_{6n+2} \\ = F_{6n+3} + F_{6n+2} + 2 (F_{6n+2} + 2 F_{6n+1}) \\ \quad \quad \quad + F_{6n+1} + F_{6n+0} \\ = F_{6n+3} + 3 F_{6n+2} + 3 F_{6n+1} + F_{6n+0} \\ = F_{6n+2} + F_{6n+1} + 3 (F_{6n+1} + F_{6n+0}) \\ \quad \quad \quad + 3 F_{6n+1} + F_{6n+0} \\ = F_{6n+2} + 7 F_{6n+1} + 4 F_{6n+0} \\ = F_{6n+1} + F_{6n+0} + 7 F_{6n+1} + 4 F_{6n+0}$$

$$F_{6n+6} = 8 F_{6n+1} + 5 F_{6n+0}$$

Hypothesis

The first hypothesis is that clearly when we decompose F_{kn+k} we will end up with

$$A \times F_{kn+1} + B \times F_{kn+0}$$

Now setting $n = 0$ gives

$$F_k = A \times F_1 + B \times F_0$$

Now as $F_1 = 1$ and $F_0 = 0$ clearly

$$F_k = k \times F_1 + B \times F_0$$

Here the value of B is immaterial

Nevertheless our second hypothesis is

$$\text{that } B = F_{k-1}$$

We therefore predict $F_7 = 13F_1 + 8F_0$

which is immediately obvious

but setting $n = 1$

$$F_{14} = 13 F_8 + 8 F_7$$

$$F_{14} = 13 \times 21 + 8 \times 13 = 377$$

a true statement.

and setting $n = 2$

$$F_{21} = 13 F_{15} + 8 F_{14}$$

$$F_{14} = 13 \times 610 + 8 \times 377 = 10946$$

a true statement

Formal Proof

$$F_{kn+k} = A \times F_{kn+1} + B \times F_{kn+0}$$

setting $n = 0$

$$F_k = A \times F_1 + B \times F_0$$

hence $A = k$

setting $n = 1$

$$F_{2k} = k \times F_{k+1} + B \times F_k$$

hence $F_{2k} \bmod F_k = 0$

Irrespective of the value of B

Hence by induction

$$F_{nk} \bmod F_k = 0 \text{ QED}$$

Narrative

If we decompose say F_{10} we can see clearly what's happening

$$\begin{aligned} F_{10} &= F_9 + F_8 \\ &= F_8 + F_7 + F_7 + F_6 \\ &= 1 F_8 + 2 F_7 + 1 F_6 \\ &= F_7 + F_6 + 2(F_6 + F_5) + F_5 + F_4 \\ &= 1 F_7 + 3 F_6 + 3 F_5 + 1 F_4 \\ &= F_6 + F_5 + 3(F_5 + F_4) + 3(F_4 + F_3) \\ &\quad + F_3 + F_2 \\ &= 1 F_6 + 4 F_5 + 6 F_4 + 4 F_3 + 1 F_2 \\ &= F_5 + F_4 + 4(F_4 + F_3) \\ &\quad + 6(F_3 + F_2) + 4(F_2 + F_1) + F_1 + F_0 \\ &= 1 F_5 + 5 F_4 + 10 F_3 + 10 F_2 + 5 F_1 + 1 F_0 \end{aligned}$$

and up to this point we are producing the terms of Pascals Triangle. But in the next decomposition the last two terms F_1 and F_0 do not decompose further.

$$\begin{aligned} F_{10} &= F_4 + F_3 + 5(F_3 + F_2) + 10(F_2 + F_1) \\ &\quad 10(F_1 + F_0) + 5F_1 + F_0 \\ &= F_4 + 6 F_3 + 15 F_2 + 25 F_1 + 11 F_0 \end{aligned}$$

and we continue with

$$F_{10} = F_3 + 7 F_2 + 46 F_1 + 26 F_0$$

$$F_{10} = F_2 + 54 F_1 + 33 F_0$$

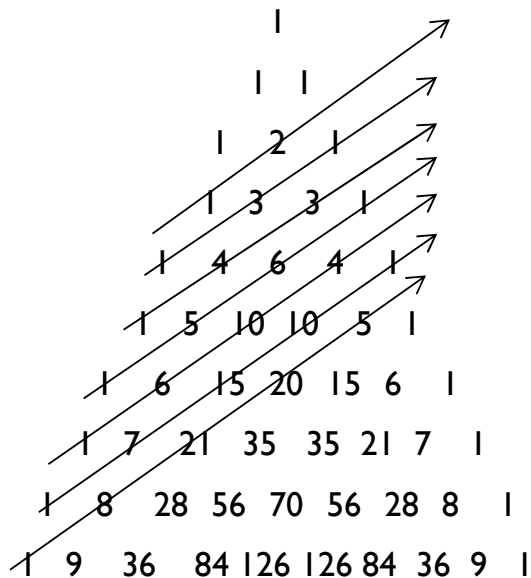
$$F_{10} = 55 F_1 + 34 F_0 \text{ as expected}$$

Thus the number of coefficients of all the terms double at each decomposition until the end terms buffer up to F_1 and F_0 and thereafter pile up until we have just F_1 and F_0 remaining.

∞ rg

Pascals and Fibonacci

The conclusion of the investigation must be that Fibonacci numbers relate to Pascal's triangle. To locate we have the terms that actually built up the final F_{10} coefficient and here they are



Numbers on the diagonal lines add to successive Fibonacci numbers.

Row Expansion

We can chart the number of terms in successive row of the expansion of F_n , and it is better to split them into odd and even rows

$2n$ 2

$4n$ 2 4 5

$6n$ 2 4 8 12 13

$8n$ 2 4 8 16 27 33 34

$10n$ 2 4 8 16 32 58 80 88 89

$3n$ 2 3

$5n$ 2 4 7 8

$7n$ 2 4 8 15 20 21

$9n$ 2 4 8 16 31 47 54 55

$11n$ 2 4 8 16 32 63 105 134 143 144

It is left as an exercise for the reader to find the general term for odd and even rows.