

An Investigation into Arithmo-Geometric Series

Derivation

Suppose I have an arithmo-geometric series of six terms

$$S_6 = a + (a + d)r + (a + 2d)r^2 + (a + 3d)r^3 + (a + 4d)r^4 + (a + 5d)r^5$$

$$= a + ar + ar^2 + ar^3 + ar^4 + ar^5 + dr + 2dr^2 + 3dr^3 + 4dr^4 + 5dr^5$$

$$rS_6 = ar + ar^2 + ar^3 + ar^4 + ar^5 + ar^6 + dr^2 + 2dr^3 + 3dr^4 + 4dr^5 + 5dr^6$$

$$S_6 - rS_6 = (a - ar^6) + dr + dr^2 + dr^3 + dr^4 + dr^5 - 5dr^6$$

$$= (a - ar^6) + d(r + r^2 + r^3 + r^4 + r^5) - 5dr^6$$

$$S_6(1 - r) = a(1 - r^6) + dr \left\{ \frac{(1 - r^5)}{(1 - r)} \right\} - 5dr^6$$

and I can see that in general

$$S_n(1 - r) = a(1 - r^n) / (1 - r) + dr(1 - r^{n-1}) / (1 - r)^2 - (n - 1)dr^n / (1 - r)$$

which doesn't look particularly simpler.

However if $r < 1$ then

$$S_\infty = a / (1 - r) + dr / (1 - r)^2$$

which is a bit neater.

Example

$$\text{Let } S_5 = 3 + 5x + 7x^2 + 9x^3 + 11x^4$$

giving $a = 3$ $d = 2$ and $r = x$. So from our formula

$$\text{Let } S_5 = 3(1 - x^5) / (1 - x) + 2x(1 - x^4) / (1 - x)^2 + 8x^5 / (1 - x)$$

which seems to move us from bad to worse. However if I enter onto my TI-83 as Y1 and Y2 I can see that the functions are equivalent except Y2 is undefined at $x = 1$ as expected. To test my infinite sum I try

$$S = 3 + \frac{5}{2} + \frac{7}{4} + \frac{9}{8} + \frac{11}{16} + \frac{13}{32} \dots$$

Summing the series to 11 terms gives 9.97.

My formula gives $3 / \frac{1}{2} + 1 / (\frac{1}{2})^2 = 10$ which seems a fair indication it's correct.

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