An Investigation into Perpendicular Distances using TI-83

Summary

This is an investigation into the perpendicular distance from a given point to a given line. There is a proof by induction of the given formula – notoriously difficult to derive algebraically – by demonstrating it is invariant to translations.

Introduction

TI-83 set Zoom/ZStandard and enter

$-10, -15, 1, -10, 15, 1, 1$

Then Zoom / 5:ZSquare

Go back to Window and observe the TI-83 rounds these values to give a square graph grid.

Enter Y1 = $^4/_3x + 8$

We will examine the perpendicular distance from point P1 (7 , 9). These values have been carefully chosen to ensure all subsequent calculations are in integers.

Now we determine a line perpendicular to Y1 passing through P1 given by

$y - y_1 = m (x - x_1)$

$y - 9 = {}^-\tfrac{3}{4} (x - 7)$

because perpendiculars $m_1 \times m_2 = {}^-1$

Therefore enter Y2 = $^-\tfrac{3}{4}\ x + {}^{57}/$

Graph Y1 and Y2 and observe perpendicular

We can find the point of intersection directly. Enter 2nd CALC 5:intersect Enter, Enter, Enter to get x=3 y=12 (3,12)

Algebraically you can deduce this by equating y values to determine x and then substituting back into Y1.

Distance P1 (7 , 9) to P2 (3 , 12) is

$d^2 = (7 - 3)^2 + (9 - 12)^2$

d = 5 *(rigged to be a 3,4,5 triangle)*

Now we translate Y1, Y2, P1 and P2 by vector $\left[{}^{-3}{}_{-12} \right]$

Y1 $\Rightarrow y + 12 = {}^4/_3(x + 3) + 8$

Enter Y3 = $^4/_3\ x$

(again the values were rigged to produce a line that passes through (0,0)

P1 $\Rightarrow$ P3 $(7 , 9) + \left[{}^{-3}{}_{-12} \right] = (4 , {}^-3)$

P2 $\Rightarrow$ P4 $(3 , 12) + \left[{}^{-3}{}_{-12} \right] = (0 , 0)$

and for completeness

Enter Y4 = $^-\tfrac{3}{4}\ x$

So now we have Y3 and Y4 intersecting at (0 , 0) and distance P3 (4 , ⁻3) to P4 (0 , 0)

$d^2 = (4 - 0)^2 + (^-3 - 0)^2$

d = 5 *as before and as expected*

Summary to Date

We took line Y1 and point P1 and calculated perpendicular distance "5"

We then translated the line and point a predetermined distance to produce an intersection at (0 , 0) checking the perpendicular distance was identical as expected. So the distance is invariant to the translation because we're translating a rigid body or network.

In principle if we could easily determine

$[\,^j{}_k]$ the calculation simplifies to

$d^2 = (x_1 + j)^2 + (y_1 + k)^2$

Repeat Algebraically in reverse

Start with line Y3 = $^p/_q x$ and point P3

The line P3 to (0 , 0) is perpendicular to Y3 because $m_1 \times m_2 = ^-1$

$d^2 = (p)^2 + (^-q)^2$

Now apply vector $[\,^j{}_k]$ to Y3 and P3

$y - k = ^p/_q (x - j)$

$y = ^p/_q (x - j) + k$ ie Y1

and P3 $\Rightarrow$ (p + j , ⁻q + k)

(0 , 0) $\Rightarrow$ (j , k) which is on Y1

So perpendicular distance between (p + j , ⁻q + k) and (j , k) is

$d^2 = (p + j - j)^2 + (^-q + k - k)^2)$

$d = \sqrt{\,} (p^2 + q^2)$ as expected.

In short the perpendicular distance d is invariant to the applied vector $[\,^j{}_k]$ to Y3 and P3.

Proof of Formula

Formula is

$\{ ax_1 + by_1 + c \} / \sqrt{\,} (a^2 + b^2)$

For y = $^p/_q$ x and point (p, ⁻q)

$d = p\times p + (-q) \times (^-q) / \sqrt{\,} \{ p^2 + (^-q^2) \}$

$= (p^2 + q^2) / \sqrt{\,} (p^2 + q^2)$

$= \sqrt{\,} (p^2 + q^2)$

We can now show the formula to be true if invariant to vector $[\,^j{}_k]$

so ax + by + c $\Rightarrow$ p(x – j) – qy + kq

ie px – qy + kq – pj = 0

and point is (p + j , k – q)

Substituting all values from formula

$d = \{ p (p + j) - q (k - q) + kq - pj \}$

$\div \sqrt{\,} \{ (p)^2 + (^-q)^2 \}$

$= \{ p^2 + pj - qk + q^2 + kq - pj \}$

$\div \sqrt{\,} \{ p^2 + q^2 \}$

$= \{ p^2 + q^2 \} / \sqrt{\,} \{ p^2 + q^2 \}$

$= \sqrt{\{ p^2 + q^2 \}}$

which is what we set out to establish.

Conclusion

We establish that

$ax_1 + by_1 + c / \sqrt{(a^2 + b^2)}$

gives the correct answer in the simplest case when the line is

$y = {}^{p}/_{q}$ and point $(p , {}^{-}q)$

ie $d = \sqrt{\{ p^2 + q^2 \}}$

We then translate the line and point by any arbitrary vector $[{}^{j}_{k}]$ and determine that the formula gives the same answer

$d = \sqrt{\{ p^2 + q^2 \}}$

Hence by induction the formula is correct.

Addendum

If you search the internet you can find about 7 proofs of the formula

$\{ ax_1 + by_1 + c \} / \sqrt{(a^2 + b^2)}$

all reasonably incomprehensible and totally non intuitive. The formula "looks" correct but emerges like a rabbit out the hat. Then I found a proof in "Elementary Calculus and Coordinate Geometry" by C G Nobbs which absolutely nails it.

Let the coordinates of point P be (x_1 , y_1) and the perpendicular to line $ax + by + c = 0$ be PN.

Let the directive angle PN be "α" and the length to be found "d".

Coordinates N are then

$(x_1 + d \cos \alpha , y_1 + d \sin \alpha)$

However this point lies on

$ax + by + c = 0$

so we substitute values for x and y.

$a (x_1 + d \cos \alpha x) +$

$b (y_1 + d \sin \alpha) + c = 0$

which we can rearrange to give

$d (a \cos \alpha + b \sin \alpha) =$

${}^{-} (ax_1 + by_1 + c)$

Gradient line is ${}^{-a}/_{b}$

so gradient PN $= {}^{b}/_{a}$

and ${}^{b}/_{a} = \tan \alpha$

so $\sin \alpha = b / \sqrt{(a^2 + b^2)}$ and

$\cos \alpha = a / \sqrt{(a^2 + b^2)}$ so

$d \{a^2 / \sqrt{(a^2 + b^2)} + b^2 / \sqrt{(a^2 + b^2)}\}$

$= {}^{-} (ax_1 + by_1 + c)$

Hence $d \sqrt{(a^2 + b^2)} =$

${}^{-} (ax_1 + by_1 + c)$

so $d = {}^{-}(ax_1 + by_1 + c) / \sqrt{(a^2 + b^2)}$

which is an intuitive process.

Investigation

Investigate the perpendicular distance from (5 , 2) to $y = {}^{-3}/_{4}\, x + 12$ from first principles and using the formula. Verify the answer from the formula is indeed ⁻5 *(negative 5)*

Explanation

If we'd taken coordinates N as

$(x_1 - d \cos \alpha , y_1 - d \sin \alpha)$ the formula gives a positive answer. Strictly it's $\pm$ and we take the modulus.

Perpendicular Form of a Line

This form can be extended. Take a line from the origin 0 (0 , 0) to point P on any line and perpendicular to it.

The coordinates P are

$$(p\cos \alpha \,.\, p \sin \alpha)$$

and the gradient is $\tan \alpha$

Therefore gradient line is ⁻cotan α

So given a point and a gradient we derive

$$(y - p \sin \alpha) =$$

$$^{-}\text{cotan}\, \alpha\, (x - p \sin \alpha)$$

Multiplying through by $\sin \alpha$ and rearranging

$$y \sin \alpha =$$

$$p (\cos^2 \alpha + \sin^2 \alpha) - x \cos \alpha$$

$$x \cos \alpha + y \sin \alpha = p$$

This is called the perpendicular form of the equation of the line

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