

Mr. G's Handbook of Mathematics

Trigonometrical (Circular) Functions

sin , *cos* , *tan* , cosec, *sec* , *cot*
and inverses

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Circular Functions - Phase Shifts

	sin	cos	tan
$\theta + 90^\circ$	$^+ \cos \theta$	$^- \sin \theta$	$^- \cot \theta$
$\theta - 90^\circ$	$^- \cos \theta$	$^+ \sin \theta$	$^- \cot \theta$
$90^\circ - \theta$	$^+ \cos \theta$	$^+ \sin \theta$	$^+ \cot \theta$
$\theta + 180^\circ$	$^- \sin \theta$	$^- \cos \theta$	$^+ \tan \theta$
$\theta - 180^\circ$	$^- \sin \theta$	$^- \cos \theta$	$^+ \tan \theta$
$180^\circ - \theta$	$^+ \sin \theta$	$^- \cos \theta$	$^- \tan \theta$
$\theta + 270^\circ$	$^- \cos \theta$	$^+ \sin \theta$	$^- \cot \theta$
$\theta - 270^\circ$	$^+ \cos \theta$	$^- \sin \theta$	$^- \cot \theta$
$270^\circ - \theta$	$^- \cos \theta$	$^- \sin \theta$	$^+ \cot \theta$
$\theta + 360^\circ$	$^+ \sin \theta$	$^+ \cos \theta$	$^+ \tan \theta$
$\theta - 360^\circ$	$^+ \sin \theta$	$^+ \cos \theta$	$^+ \tan \theta$
$360^\circ - \theta$	$^- \sin \theta$	$^+ \cos \theta$	$^- \tan \theta$
$^- \theta$	$^- \sin \theta$	$^+ \cos \theta$	$^- \tan \theta$
$^- \theta$	$^- \operatorname{cosec} \theta$	$^+ \sec \theta$	$^- \cot \theta$
functions are	odd	even	odd

Hyperbolic Functions - Sign Changes

$^- x$	$^- \sinh x$	$^+ \cosh x$	$^- \tanh x$
$^- x$	$^- \operatorname{cosech} x$	$^+ \operatorname{sech} x$	$^- \coth x$
functions are	odd	even	odd

Notes

The sign changes for **cosine** and **sine** cover all the 8 permutations of + and -

cosines are even functions ($\cos \theta = \cos ^- \theta$), **sines** are odd functions ($\sin \theta = ^- \sin ^- \theta$)

Circular Function Values

		sin		cos		tan	
$-\pi$	-180°	0	$-\sqrt{0}/2$	-1	$-\sqrt{4}/2$	0	$\sqrt{0}/\sqrt{4}$
$-5\pi/6$	-150°	$-1/2$	$-\sqrt{1}/2$	$-\sqrt{3}/2$	$-\sqrt{3}/2$	$\sqrt{3}/3$	$\sqrt{1}/\sqrt{3}$
$-3\pi/4$	-135°	$-\sqrt{2}/2$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	1	$\sqrt{2}/\sqrt{2}$
$-2\pi/3$	-120°	$-\sqrt{3}/2$	$-\sqrt{3}/2$	$-1/2$	$-\sqrt{1}/2$	$\sqrt{3}$	$\sqrt{3}/\sqrt{1}$
$-\pi/2$	-90°	-1	$-\sqrt{4}/2$	0	$\sqrt{0}/2$	$+\infty$	$-\sqrt{4}/\sqrt{0}$
$-\pi/3$	-60°	$-\sqrt{3}/2$	$-\sqrt{3}/2$	$1/2$	$\sqrt{1}/2$	$-\sqrt{3}$	$-\sqrt{3}/\sqrt{1}$
$-\pi/4$	-45°	$-\sqrt{2}/2$	$-\sqrt{2}/2$	$\sqrt{2}/2$	$\sqrt{2}/2$	-1	$-\sqrt{2}/\sqrt{2}$
$-\pi/6$	-30°	$-1/2$	$-\sqrt{1}/2$	$\sqrt{3}/2$	$\sqrt{3}/2$	$-\sqrt{3}/3$	$-\sqrt{1}/\sqrt{3}$
0	0°	0	$\sqrt{0}/2$	1	$\sqrt{4}/2$	0	$\sqrt{0}/\sqrt{4}$
$\pi/12$	15°	$1/4(\sqrt{6}-\sqrt{2})$		$1/4(\sqrt{6}+\sqrt{2})$		$2-\sqrt{3}$	
$\pi/6$	30°	$1/2$	$\sqrt{1}/2$	$\sqrt{3}/2$	$\sqrt{3}/2$	$\sqrt{3}/3$	$\sqrt{1}/\sqrt{3}$
$\pi/5$	36°	$1/2\sqrt{\{1/2(5-\sqrt{5})\}}$		$1/4(1+\sqrt{5})$			
$\pi/4$	45°	$\sqrt{2}/2$	$\sqrt{2}/2$	$\sqrt{2}/2$	$\sqrt{2}/2$	1	$\sqrt{2}/\sqrt{2}$
$3\pi/10$	54°	$1/4(1+\sqrt{5})$		$1/2\sqrt{\{1/2(5-\sqrt{5})\}}$			
$\pi/3$	60°	$\sqrt{3}/2$	$\sqrt{3}/2$	$1/2$	$\sqrt{1}/2$	$\sqrt{3}$	$\sqrt{3}/\sqrt{1}$
$5\pi/12$	75°	$1/4(\sqrt{6}+\sqrt{2})$		$1/4(\sqrt{6}-\sqrt{2})$		$2+\sqrt{3}$	
$\pi/2$	90°	1	$\sqrt{4}/2$	0	$\sqrt{0}/2$	$+\infty$	$\sqrt{4}/\sqrt{0}$
$2\pi/3$	120°	$\sqrt{3}/2$	$\sqrt{3}/2$	$-1/2$	$-\sqrt{1}/2$	$-\sqrt{3}$	$-\sqrt{3}/\sqrt{1}$
$3\pi/4$	135°	$\sqrt{2}/2$	$\sqrt{2}/2$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	-1	$-\sqrt{2}/\sqrt{2}$
$5\pi/6$	150°	$1/2$	$\sqrt{1}/2$	$-\sqrt{3}/2$	$-\sqrt{3}/2$	$-\sqrt{3}/3$	$-\sqrt{1}/\sqrt{3}$
π	180°	0	$-\sqrt{0}/2$	-1	$-\sqrt{4}/2$	0	$\sqrt{0}/\sqrt{4}$
$7\pi/6$	210°	$-1/2$	$-\sqrt{1}/2$	$-\sqrt{3}/2$	$-\sqrt{3}/2$	$\sqrt{3}/3$	$\sqrt{1}/\sqrt{3}$
$5\pi/4$	225°	$-\sqrt{2}/2$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	1	$\sqrt{2}/\sqrt{2}$
$4\pi/3$	240°	$-\sqrt{3}/2$	$-\sqrt{3}/2$	$-1/2$	$-\sqrt{1}/2$	$\sqrt{3}$	$\sqrt{3}/\sqrt{1}$
$3\pi/2$	270°	-1	$-\sqrt{4}/2$	0	$\sqrt{0}/2$	$+\infty$	$-\sqrt{4}/\sqrt{0}$
$5\pi/3$	300°	$-\sqrt{3}/2$	$-\sqrt{3}/2$	$1/2$	$\sqrt{1}/2$	$-\sqrt{3}$	$-\sqrt{3}/\sqrt{1}$
$7\pi/4$	315°	$-\sqrt{2}/2$	$-\sqrt{2}/2$	$\sqrt{2}/2$	$\sqrt{2}/2$	-1	$-\sqrt{2}/\sqrt{2}$
$11\pi/6$	330°	$-1/2$	$-\sqrt{1}/2$	$\sqrt{3}/2$	$\sqrt{3}/2$	$-\sqrt{3}/3$	$-\sqrt{1}/\sqrt{3}$
2π	360°	0	$-\sqrt{0}/2$	1	$\sqrt{4}/2$	0	$-\sqrt{0}/\sqrt{4}$

Basic Trigonometric (Circular) Relationships

$$\begin{aligned} 1 &= \cos^2 \theta + \sin^2 \theta \\ \sec^2 \theta &= 1 + \tan^2 \theta & \operatorname{cosec}^2 \theta &= 1 + \cot^2 \theta \end{aligned}$$

Addition Formulae

$$\begin{aligned} \sin(\theta \pm \phi) &= \sin \theta \cos \phi \pm \cos \theta \sin \phi \\ \cos(\theta \pm \phi) &= \cos \theta \cos \phi \mp \sin \theta \sin \phi \end{aligned}$$

Double Angle Formulae ¹

$$\begin{aligned} \sin 2\theta &= 2 \sin \theta \cos \theta \\ \cos 2\theta &= 1 - 2 \sin^2 \theta \\ \cos 2\theta &= 2 \cos^2 \theta - 1 \\ \cos 2\theta &= \cos^2 \theta - \sin^2 \theta \end{aligned}$$

Half Angle Formulae

$$\begin{aligned} \sin^2 \theta &= \frac{1}{2}(1 - \cos 2\theta) \\ \cos^2 \theta &= \frac{1}{2}(1 + \cos 2\theta) \\ \tan^2 \theta &= \frac{1 - \cos 2\theta}{1 + \cos 2\theta} \\ \cos 2\theta &= \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \end{aligned}$$

Product Formulae ²

$$\begin{aligned} \sin \theta \cos \phi &= \frac{1}{2} [\sin(\theta + \phi) + \sin(\theta - \phi)] \\ \cos \theta \sin \phi &= \frac{1}{2} [\sin(\theta + \phi) - \sin(\theta - \phi)] \text{ because } \sin^{-\theta} = -\sin \theta \\ \cos \theta \cos \phi &= \frac{1}{2} [\cos(\theta + \phi) + \cos(\theta - \phi)] \\ \sin \theta \sin \phi &= -\frac{1}{2} [\cos(\theta + \phi) - \cos(\theta - \phi)] \end{aligned}$$

Factor / Sum Formulae

$$\begin{aligned} \text{set } \theta + \phi &= P & \text{set } \theta - \phi &= Q \\ \text{then } \theta &= \frac{1}{2}(P + Q) & \text{then } \phi &= \frac{1}{2}(P - Q) \\ \sin P + \sin Q &= 2 \sin \left[\frac{1}{2}(P + Q) \right] \cos \left[\frac{1}{2}(P - Q) \right] \text{ }^3 \\ \sin P - \sin Q &= 2 \cos \left[\frac{1}{2}(P + Q) \right] \sin \left[\frac{1}{2}(P - Q) \right] \\ \cos P + \cos Q &= 2 \cos \left[\frac{1}{2}(P + Q) \right] \cos \left[\frac{1}{2}(P - Q) \right] \\ \cos P - \cos Q &= -2 \sin \left[\frac{1}{2}(P + Q) \right] \sin \left[\frac{1}{2}(P - Q) \right] \end{aligned}$$

Notes

¹ For further insight into asymmetry between **sine** and **cosine** see section on power series

² These are useful for integration but were originally derived by Viète as a precursor to logs.

³ This form gives the resultant addition of two frequencies

Trigonometrical (Circular) Relationships 2

$$\text{define } \tan \theta = \frac{\sin \theta}{\cos \theta} \quad \tan \left(\frac{1}{2}\pi - \phi \right) = \cot \theta$$

$$\tan (\theta + \phi) = \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi} \quad \text{try setting } \theta = \frac{1}{4}\pi$$

$$\tan (\theta - \phi) = \frac{\tan \theta - \tan \phi}{1 + \tan \theta \tan \phi}$$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$\sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta} \quad \text{from } \sec^2 \theta = 1 + \tan^2 \theta$$

$$\cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$$

$$\cot (\theta + \phi) = \frac{\cot \phi \cdot \cot \theta - 1}{\cot \phi + \cot \theta}$$

$$\cot (\theta - \phi) = \frac{\cot \phi \cot \theta + 1}{\cot \phi - \cot \theta}$$

$$\cot 2\theta = \frac{\cot^2 \theta - 1}{2 \cot \theta}$$

$$\cot \theta = \operatorname{cosec} 2\theta + \cot 2\theta$$

$$\tan \theta = \operatorname{cosec} 2\theta - \cot 2\theta$$

$$\cot \left(\theta - \frac{1}{2}\pi \right) = \cot 2\theta - \operatorname{cosec} 2\theta \quad \text{cf } \int \operatorname{cosec} \theta \, d\theta = -\tan \theta$$

$$\tan \theta = \frac{1}{\operatorname{cosec} 2\theta + \cot 2\theta}$$

$$\cot \theta = \frac{1}{\operatorname{cosec} 2\theta - \cot 2\theta}$$

$$\tan \left(\theta - \frac{1}{2}\pi \right) = \frac{1}{\cot 2\theta - \operatorname{cosec} 2\theta} = -\cot \theta$$

$$\tan \theta = \frac{\sin 2\theta}{1 + \cos 2\theta}$$

$$\cot \theta = \frac{\sin 2\theta}{1 - \cos 2\theta}$$

$$\tan \left(\theta - \frac{1}{2}\pi \right) = \frac{\sin 2\theta}{\cos 2\theta - 1} = -\cot \theta$$

$$\cot \left(\theta + \frac{1}{4}\pi \right) = \frac{\cos 2\theta}{1 + \sin 2\theta}$$

$$\tan \left(\theta + \frac{1}{4}\pi \right) = \frac{\cos 2\theta}{1 - \sin 2\theta}$$

$$\cot \left(\theta - \frac{1}{4}\pi \right) = \frac{\cos 2\theta}{\sin 2\theta - 1}$$

$$a \sin \theta + b \cos \theta = R \sin (\theta + \phi) \quad 0 < \alpha < 90^\circ \text{ or } (\frac{1}{2} \pi)$$

$$a \cos \theta + b \sin \theta = R \cos (\theta - \phi)$$

$$\text{where } R \cos \phi = a \quad \text{and } R \sin \phi = b$$

$$R = \sqrt{a^2 + b^2}$$

$$\text{and } \phi = \tan^{-1} b/a$$

Trigonometrical (Circular) Relationships 3

$$\begin{aligned}
 (1 + \sin \theta) / (1 - \sin \theta) &= (\sec \theta + \tan \theta)^2 &= \tan^2 (\tfrac{1}{2}\theta + \tfrac{1}{4}\pi) \\
 (1 - \sin \theta) / (1 + \sin \theta) &= (\sec \theta - \tan \theta)^2 &= \cot^2 (\tfrac{1}{2}\theta + \tfrac{1}{4}\pi) \\
 (\sin \theta + 1) / (\sin \theta - 1) &= -(\sec \theta + \tan \theta)^2 &= \cot^2 (\tfrac{1}{2}\theta - \tfrac{1}{4}\pi) \\
 (\sin \theta - 1) / (\sin \theta + 1) &= -(\sec \theta - \tan \theta)^2 &= \tan^2 (\tfrac{1}{2}\theta - \tfrac{1}{4}\pi) \\
 (1 + \cos \theta) / (1 - \cos \theta) &= (\operatorname{cosec} \theta + \cot \theta)^2 &= \cot^2 (\tfrac{1}{2}\theta) \\
 (1 - \cos \theta) / (1 + \cos \theta) &= (\operatorname{cosec} \theta - \cot \theta)^2 &= \tan^2 (\tfrac{1}{2}\theta) \\
 (\cos \theta + 1) / (\cos \theta - 1) &= -(\operatorname{cosec} \theta + \cot \theta)^2 &= \tan^2 (\tfrac{1}{2}\theta - \tfrac{1}{2}\pi) \\
 (\cos \theta - 1) / (\cos \theta + 1) &= -(\operatorname{cosec} \theta - \cot \theta)^2 &= \cot^2 (\tfrac{1}{2}\theta - \tfrac{1}{2}\pi) \\
 (1 + \tan \theta) / (1 - \tan \theta) &= \sec 2\theta + \tan 2\theta &= \tan (\theta + \tfrac{1}{4}\pi) \\
 (1 - \tan \theta) / (1 + \tan \theta) &= \sec 2\theta - \tan 2\theta &= \cot (\theta + \tfrac{1}{4}\pi) \\
 (\tan \theta + 1) / (\tan \theta - 1) &= -(\sec 2\theta + \tan 2\theta) &= \cot (\theta - \tfrac{1}{4}\pi) \\
 (\tan \theta - 1) / (\tan \theta + 1) &= -(\sec 2\theta - \tan 2\theta) &= \tan (\theta - \tfrac{1}{4}\pi)
 \end{aligned}$$

Inverse Circular Functions I

$$\begin{aligned}
 \sin^{-1} x &= -i \ln (ix \pm \sqrt{1 - x^2}) & -1 \leq x \leq +1 \\
 \sin^{-1} x &= -i \ln (ix \pm i\sqrt{x^2 - 1}) & -1 \leq x \leq +1 \\
 \operatorname{cosec}^{-1} x &= -i \ln (i/x \pm \sqrt{1 - 1/x^2}) & x \leq -1 \text{ and } +1 \leq x \\
 \cos^{-1} x &= -i \ln (x \pm \sqrt{x^2 - 1}) & -1 \leq x \leq +1 \\
 \cos^{-1} x &= -i \ln (x \pm i\sqrt{1 - x^2}) & -1 \leq x \leq +1 \\
 \sec^{-1} x &= -i \ln (1/x \pm \sqrt{1/x^2 - 1}) & x \leq -1 \text{ and } +1 \leq x \\
 \tan^{-1} x &= \tfrac{1}{2} i \ln (i^{+x}/i - x) & x \in \mathbb{P} \\
 \cot^{-1} x &= \tfrac{1}{2} i \ln (i^{x+1}/i x - 1) & x \in \mathbb{P}
 \end{aligned}$$

Notes

Inverse circular functions used to be termed "goniometric". They are many-valued.

Inverse Circular Functions 2

$$\text{if } \sin b = 1/a$$

$$\text{then } \operatorname{cosec} b = a$$

$$\text{hence } b = \sin^{-1}(1/a)$$

$$\text{and also } b = \operatorname{cosec}^{-1} a$$

$$\text{hence } \operatorname{cosec}^{-1}(a) = \sin^{-1}(1/a)$$

$\sin^{-1}x$ is an odd function

$$\sin^{-1}(-x) = -\sin^{-1}x \quad -1 \leq x \leq +1$$

$$\operatorname{cosec}^{-1}x = \sin^{-1}(1/x) \quad x \leq -1 \text{ and } +1 \leq x$$

$(\cos^{-1}x - \pi/2)$ is an odd function

$$\cos^{-1}(-x) = -\cos^{-1}x + \pi \quad -1 \leq x \leq +1$$

$$\sec^{-1}x = \cos^{-1}(1/x) \quad x \leq -1 \text{ and } +1 \leq x$$

$\tan^{-1}x$ is an odd function

$$\tan^{-1}(-x) = -\tan^{-1}x \quad x \in \mathbb{P}$$

$$\cot^{-1}x = \tan^{-1}(1/x) \quad x \in \mathbb{P}$$

Inverse Circular Functions 3

$$\sin^{-1}x + \cos^{-1}x = 1/2 \pi \quad -1 \leq x \leq +1$$

$$\operatorname{cosec}^{-1}x + \sec^{-1}x = 1/2 \pi \quad x \leq -1 \text{ and } +1 \leq x$$

$$\tan^{-1}x + \cot^{-1}x = 1/2 \pi \quad x < 0 \quad \dagger$$

$$\tan^{-1}x + \cot^{-1}x = 1/2 \pi \quad x > 0 \quad \dagger$$

$$\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x \quad x \leq -1 \text{ and } +1 \leq x$$

$$\sec^{-1}(-x) = \pi - \sec^{-1}x \quad x \leq -1 \text{ and } +1 \leq x$$

$$\cot^{-1}(-x) = -\cot^{-1}x \quad x \in \mathbb{P} \quad x \neq 0$$

Notes

$$\tan^{-1}x \pm \tan^{-1}y = \tan^{-1}\{(x \pm y) / (1 \mp xy)\}$$

Inverse circular functions give the arc length of a sector on the unit circle $x^2 + y^2 = 1$

This is why the functions may also be written **arcsin**, **arccos**, **arctan** etc.

But note it's **arsinh**, **arcosh**, **artanh** etc. because this function relates to area

Trigonometric (Circular) Power Series Basic Relationship

$$z^n + 1/z^n = 2 \cos n\theta \qquad z^n - 1/z^n = 2i \sin n\theta \quad (z^n = \text{cis } n\theta)$$

even powers $\cos^n$ in cosines Expand $(z + 1/z)^n$ for n even

$$\cos^2 \theta = 1/2^1 (1 \cos 2\theta + 1)$$

$$\cos^4 \theta = 1/2^3 (1 \cos 4\theta + 4 \cos 2\theta + 3)$$

$$\cos^6 \theta = 1/2^5 (1 \cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10)$$

$$\cos^8 \theta = 1/2^7 (1 \cos 8\theta + 8 \cos 6\theta + 28 \cos 4\theta + 56 \cos 2\theta + 35)$$

odd powers $\cos^n$ in cosines Expand $(z + 1/z)^n$ for n odd

$$\cos^3 \theta = 1/2^2 (1 \cos 3\theta + 3 \cos \theta)$$

$$\cos^5 \theta = 1/2^4 (1 \cos 5\theta + 5 \cos 3\theta + 10 \cos \theta)$$

$$\cos^7 \theta = 1/2^6 (1 \cos 7\theta + 7 \cos 5\theta + 21 \cos 3\theta + 35 \cos \theta)$$

even powers $\sin^n$ in cosines Expand $(z - 1/z)^n$ for n even

$$\sin^2 \theta = 1/2^1 (1 - 1 \cos 2\theta)$$

$$\sin^4 \theta = 1/2^3 (3 - 4 \cos 2\theta + 1 \cos 4\theta)$$

$$\sin^6 \theta = 1/2^5 (10 - 15 \cos 2\theta + 6 \cos 4\theta - 1 \cos 6\theta)$$

$$\sin^8 \theta = 1/2^7 (35 - 56 \cos 2\theta + 28 \cos 4\theta - 8 \cos 6\theta + 1 \cos 8\theta)$$

odd powers $\sin^n$ in sines Expand $(z - 1/z)^n$ for n odd

$$\sin^3 \theta = 1/2^2 (\sin 3\theta - 3 \sin \theta)$$

$$\sin^5 \theta = 1/2^4 (\sin 5\theta - 5 \sin 3\theta + 10 \sin \theta)$$

$$\sin^7 \theta = 1/2^6 (\sin 7\theta - 7 \sin 5\theta + 21 \sin 3\theta - 35 \sin \theta)$$

Notes

There is no apparent way to produce the following series expansions

even powers $\cos^n$ in sines

odd powers $\cos^n$ in sines

even powers $\sin^n$ in sines

odd powers $\sin^n$ in cosines

and this shows why 3 of the 4 double angle formulae are **cos** and only one for **sin**

Expansion $\cos(n\theta)$ and $\sin(n\theta)$ in power series of $\cos \theta$ and $\sin \theta$

from de Moivre's theorem $(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta)$ we can create power series by expanding $(\cos \theta + i \sin \theta)$ and equating real and imaginary terms.

expansion $\cos(n\theta)$ for n odd *Coefficients are Chebyshev Polynomials $T_n(x)$*

$$\begin{aligned}\cos \theta &= 1 \cos \theta \\ \cos 3\theta &= 4 \cos^3 \theta - 3 \cos \theta \\ \cos 5\theta &= 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta \\ \cos 7\theta &= 64 \cos^7 \theta - 112 \cos^5 \theta + 56 \cos^3 \theta - 7 \cos \theta\end{aligned}$$

expansion $\cos(n\theta)$ for n even *Coefficients are Chebyshev Polynomials $T_n(x)$*

$$\begin{aligned}\cos 2\theta &= 2 \cos^2 \theta - 1 \\ \cos 4\theta &= 8 \cos^4 \theta - 8 \cos^2 \theta + 1 \\ \cos 6\theta &= 32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1 \\ \cos 8\theta &= 128 \cos^8 \theta - 256 \cos^6 \theta + 160 \cos^4 \theta - 32 \cos^2 \theta + 1\end{aligned}$$

expansion $\sin(n\theta)$ for n odd *nb cf coefficients with $\cos$ expansions.*

$$\begin{aligned}\sin \theta &= 1 \sin \theta \\ \sin 3\theta &= 3 \sin \theta - 4 \sin^3 \theta \\ \sin 5\theta &= 5 \sin \theta - 20 \sin^3 \theta + 16 \sin^5 \theta \\ \sin 7\theta &= 7 \sin \theta - 56 \sin^3 \theta + 112 \sin^5 \theta - 64 \sin^7 \theta\end{aligned}$$

expansion $\sin(n\theta)$ for n even *nb annoying $(\sin \theta \cos \theta)$ term recurs.*

$$\begin{aligned}\sin 2\theta &= 2 \sin \theta \cos \theta (1) \\ \sin 4\theta &= 2 \sin \theta \cos \theta (-4 \sin^2 \theta + 2) \\ \sin 6\theta &= 2 \sin \theta \cos \theta (16 \sin^4 \theta - 16 \sin^2 \theta + 3) \\ \sin 8\theta &= 2 \sin \theta \cos \theta (-64 \sin^6 \theta + 96 \sin^4 \theta - 40 \sin^2 \theta + 4)\end{aligned}$$

Notes

$T_n(x)$ is generated by $T_0(x) = 1$; $T_1(x) = x$ and thereafter $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$

Further Circular Function Values

		sin		cos	
0	0°	0		1	
$\pi/10$	18°	$\frac{1}{4}(\sqrt{5} - 1)$	$\frac{1}{2}(\phi - 1)$	$\frac{1}{4}\sqrt{(10+2\sqrt{5})}$	$\phi(\sqrt{(1-\frac{1}{4}\phi^2)})$
$\pi/5$	36°	$\frac{1}{2}\sqrt{(\frac{1}{2}(5-\sqrt{5}))}$	$(\sqrt{(1-\frac{1}{4}\phi^2)})$	$\frac{1}{4}(\sqrt{5} + 1)$	$\frac{1}{2}(\phi)$
$3\pi/10$	54°	$\frac{1}{4}(\sqrt{5} + 1)$	$\frac{1}{2}(\phi)$	$\frac{1}{2}\sqrt{(\frac{1}{2}(5-\sqrt{5}))}$	$(\sqrt{(1-\frac{1}{4}\phi^2)})$
$2\pi/5$	72°	$\frac{1}{4}\sqrt{(10+2\sqrt{5})}$	$\phi(\sqrt{(1-\frac{1}{4}\phi^2)})$	$\frac{1}{4}(\sqrt{5} - 1)$	$\frac{1}{2}(\phi - 1)$
$\pi/2$	90°	1		0	
$3\pi/5$	108°	$\frac{1}{4}\sqrt{(10+2\sqrt{5})}$	$\phi(\sqrt{(1-\frac{1}{4}\phi^2)})$	$-\frac{1}{4}(\sqrt{5} - 1)$	$-\frac{1}{2}(\phi - 1)$
$7\pi/10$	126°	$\frac{1}{4}(\sqrt{5} + 1)$	$\frac{1}{2}(\phi)$	$-\frac{1}{2}\sqrt{(\frac{1}{2}(5-\sqrt{5}))}$	$-(\sqrt{(1-\frac{1}{4}\phi^2)})$
$4\pi/5$	144°	$\frac{1}{2}\sqrt{(\frac{1}{2}(5-\sqrt{5}))}$	$(\sqrt{(1-\frac{1}{4}\phi^2)})$	$-\frac{1}{4}(\sqrt{5} + 1)$	$-\frac{1}{2}(\phi)$
$9\pi/10$	162°	$\frac{1}{4}(\sqrt{5} - 1)$	$\frac{1}{2}(\phi - 1)$	$-\frac{1}{4}\sqrt{(10+2\sqrt{5})}$	$-\phi(\sqrt{(1-\frac{1}{4}\phi^2)})$
π	180°	0		-1	

Notes on Fibonacci Series

The golden ratio $\phi = \frac{1}{2}(1 + \sqrt{5})$ also equal to $\frac{\sin 72^\circ}{\sin 36^\circ}$

ϕ is the limit of the ratios of adjacent terms of the Fibonacci series

$\phi^a - (1 - \phi^a) / \sqrt{5}$ gives the a^{th} Fibonacci number.

The first 15 Fibonacci numbers are 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377

Remember every sine and cosine value are related by the equation $\sin^2 + \cos^2 = 1$

Demonstrating this algebraically at values such as $2\pi/5$ is actually a bit tricky.

The insight into all this lies in examining the 5th roots of 1 using complex numbers

Further Notes on Expansion of $\cos n\theta$ and $\sin n\theta$

Expansion of $\cos n\theta$

$$\cos n\theta = \cos^n \theta - \frac{n(n-1)}{2!} \cos^{n-2} \theta \sin^2 \theta + \frac{n(n-1)(n-2)(n-3)}{4!} \cos^{n-4} \theta \sin^4 \theta \dots$$

Expansion of $\sin n\theta$

$$\sin n\theta = n \cos^{n-1} \theta \sin \theta - \frac{n(n-1)(n-2)}{3!} \cos^{n-3} \theta \sin^3 \theta + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!} \cos^{n-5} \theta \sin^5 \theta \dots$$

Each expansion continues until one term of $n(n-1)(n-2) \dots$ equates to zero

Circular "Functions of Functions" Mapping

$\cos(\cos^{-1} x)$	$= x$	$\sec(\cos^{-1})$	$= 1/x$
$\sec(\sec^{-1} x)$	$= x$	$\cos(\sec^{-1} x)$	$= 1/x$
$\sin(\sin^{-1} x)$	$= x$	$\operatorname{cosec}(\sin^{-1} x)$	$= 1/x$
$\operatorname{cosec}(\operatorname{cosec}^{-1} x)$	$= x$	$\sin(\operatorname{cosec}^{-1} x)$	$= 1/x$
$\tan(\tan^{-1} x)$	$= x$	$\cot(\tan^{-1} x)$	$= 1/x$
$\cot(\cot^{-1} x)$	$= x$	$\tan(\cot^{-1} x)$	$= 1/x$
$\sec(\tan^{-1} x)$	$= \sqrt{x^2 + 1}$	$\cos(\tan^{-1} x)$	$= 1/\sqrt{x^2 + 1}$
$\operatorname{cosec}(\cot^{-1} x)$	$= \sqrt{x^2 + 1}$	$\sin(\cot^{-1} x)$	$= 1/\sqrt{x^2 + 1}$
$\sec(\cot^{-1})$	$= \sqrt{x^2 + 1}/x$	$\cos(\cot^{-1})$	$= x/\sqrt{x^2 + 1}$
$\operatorname{cosec}(\tan^{-1} x)$	$= \sqrt{x^2 + 1}/x$	$\sin(\tan^{-1} x)$	$= x/\sqrt{x^2 + 1}$
$\tan(\sec^{-1} x)$	$= \sqrt{x^2 - 1}$	$\cot(\sec^{-1} x)$	$= 1/\sqrt{x^2 - 1}$
$\cot(\operatorname{cosec}^{-1} x)$	$= \sqrt{x^2 - 1}$	$\tan(\operatorname{cosec}^{-1} x)$	$= 1/\sqrt{x^2 - 1}$
$\cos(\operatorname{cosec}^{-1} x)$	$= \sqrt{x^2 - 1}/x$	$\sec(\operatorname{cosec}^{-1} x)$	$= x/\sqrt{x^2 - 1}$
$\sin(\sec^{-1} x)$	$= \sqrt{x^2 - 1}/x$	$\operatorname{cosec}(\sec^{-1} x)$	$= x/\sqrt{x^2 - 1}$
$\cos(\sin^{-1} x)$	$= \sqrt{1 - x^2}$	$\sec(\sin^{-1} x)$	$= 1/\sqrt{1 - x^2}$
$\sin(\cos^{-1} x)$	$= \sqrt{1 - x^2}$	$\operatorname{cosec}(\cos^{-1} x)$	$= 1/\sqrt{1 - x^2}$
$\tan(\cos^{-1} x)$	$= \sqrt{1 - x^2}/x$	$\cot(\cos^{-1} x)$	$= x/\sqrt{1 - x^2}$
$\cot(\sin^{-1} x)$	$= \sqrt{1 - x^2}/x$	$\tan(\sin^{-1} x)$	$= x/\sqrt{1 - x^2}$

Notes

Taking the functions $\sin$ $\cos$ $\tan$ and cosec $\sec$ $\cot$ we can perm into 36 relationships

If we arrange these by the result rather than the relationship the following pattern emerges

Personally I got a great deal of satisfaction in structuring this which took some time!

Functions of Functions

$$\begin{array}{ll}
 \sin(\cos^{-1} x/a) = \sqrt{(a^2 - x^2)}/|a| & \sin(\sec^{-1} x/a) = \sqrt{(x^2 - a^2)}/|x| \\
 \sin(\tan^{-1} x/a) = x \operatorname{sign}(a)/\sqrt{(x^2 + a^2)} & \sin(\cot^{-1} x/a) = |a|/\sqrt{(x^2 + a^2)} \\
 \cos(\sin^{-1} x/a) = \sqrt{(a^2 - x^2)}/|a| & \cos(\operatorname{cosec}^{-1} x/a) = \sqrt{(x^2 - a^2)}/|x| \\
 \cos(\tan^{-1} x/a) = |a|/\sqrt{(x^2 + a^2)} & \cos(\cot^{-1} x/a) = x \operatorname{sign}(a)/\sqrt{(x^2 + a^2)} \\
 \tan(\sin^{-1} x/a) = x \operatorname{sign}(a)/\sqrt{(a^2 - x^2)} & \tan(\operatorname{cosec}^{-1} x/a) = a \operatorname{sign}(x)/\sqrt{(x^2 - a^2)} \\
 \tan(\cos^{-1} x/a) = \sqrt{(a^2 - x^2)} \operatorname{sign}(a)/x & \tan(\sec^{-1} x/a) = \sqrt{(x^2 - a^2)} \operatorname{sign}(x)/a
 \end{array}$$

For functions **cosec**, **sec** and **cot** just invert the corresponding function **sin**, **cos** and **tan**.

For functions **cosec**⁻¹, **sec**⁻¹ and **cot**⁻¹, coeff. **x** and **a** change places. (1 exception)

Further Identities

$$(\sin \theta + \cos \theta)^2 = 1 + \sin 2\theta$$

$$(\sin \theta - \cos \theta)^2 = 1 - \sin 2\theta$$

$$\cos(\theta + \phi) \cos(\theta - \phi) = \cos^2 \theta - \sin^2 \phi$$

$$\sin(\theta + \phi) \sin(\theta - \phi) = \cos^2 \phi - \cos^2 \theta$$

$$\text{From } \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

we can derive the following relations

$$\tan \frac{1}{2}\theta = t$$

$$\sin \theta = \frac{2t}{1 + t^2}$$

$$\frac{d\theta}{dt} = \frac{2}{1 + t^2}$$

$$\cos \theta = \frac{1 - t^2}{1 + t^2}$$

$$\tan \theta = \frac{2t}{1 - t^2}$$

General Solutions

$$\text{for } \sin x = \sin \alpha \quad \text{then} \quad x = n\pi + (-1)^n \alpha$$

$$\text{for } \cos x = \cos \alpha \quad \text{then} \quad x = 2n\pi \pm \alpha$$

$$\text{for } \tan x = \tan \alpha \quad \text{then} \quad x = n\pi + \alpha$$

Domain and Range of Trigonometric Functions

Function	Domain	Range	Type	Exceptions
$\sin \theta$	$\theta \in \mathbb{P}$	$-1 \leq y \leq +1$	odd function	
$\cos \theta$	$\theta \in \mathbb{P}$	$-1 \leq y \leq +1$	even function	
$\tan \theta$	$\theta \in \mathbb{P}$	$y \in \mathbb{P}$	odd function	$x \neq \pm \frac{1}{2}\pi \pm 2n\pi$
$\operatorname{cosec} \theta$	$\theta \in \mathbb{P}$	$y \leq -1 \quad +1 \leq y$	odd function	$x \neq \pm \pi \pm 2n\pi$
$\sec \theta$	$\theta \in \mathbb{P}$	$y \leq -1 \quad +1 \leq y$	even function	$x \neq \pm \frac{1}{2}\pi \pm 2n\pi$
$\cot \theta$	$\theta \in \mathbb{P}$	$y \in \mathbb{P}$	odd function	$x \neq \pm \pi \pm 2n\pi$
$\sin^{-1} x$	$-1 \leq x \leq +1$	$-\frac{1}{2}\pi \leq y \leq +\frac{1}{2}\pi$	odd function	
$\cos^{-1} x$	$-1 \leq x \leq +1$	$0 \leq y \leq +\pi$	nb $(\cos^{-1} x - \pi)$ is an odd function	
$\tan^{-1} x$	$x \in \mathbb{P}$	$-\frac{1}{2}\pi < y < +\frac{1}{2}\pi$	odd function	
$\operatorname{cosec}^{-1} x$	$x \leq -1 \quad +1 \leq x$	$-\frac{1}{2}\pi \leq y \leq +\frac{1}{2}\pi$	odd function	
$\sec^{-1} x$	$x \leq -1 \quad +1 \leq x$	$0 \leq y \leq +\pi$	nb $(\sec^{-1} x - \pi)$ is an odd function	
$\cot^{-1} x$	$x \in \mathbb{P}$	$-\frac{1}{2}\pi < y < +\frac{1}{2}\pi$	odd function	$x \neq 0$
$\sinh x$	$x \in \mathbb{P}$	$y \in \mathbb{P}$	odd function	
$\cosh x$	$x \in \mathbb{P}$	$+1 \leq y$	even function	A catenary curve
$\tanh x$	$x \in \mathbb{P}$	$-1 < y < +1$	odd function	
$\operatorname{cosech} x$	$x \in \mathbb{P}$	$y \in \mathbb{P}$	odd function	$x \neq 0 \quad y \neq 0$
$\operatorname{sech} x$	$x \in \mathbb{P}$	$0 < y \leq +1$	even function [†]	max. at $x = 0$
$\coth x$	$x \in \mathbb{P}$	$y < -1 \quad +1 < y$	odd function	$x \neq 0$
$\sinh^{-1} x$	$x \in \mathbb{P}$	$y \in \mathbb{P}$	odd function	
$\cosh^{-1} x$	$1 < x$	$0 \leq y$		
$\tanh^{-1} x$	$-1 < x < +1$	$-\pi < y < +\pi$	odd function	
$\operatorname{cosech}^{-1} x$	$x \in \mathbb{P}$	$y \in \mathbb{P}$	odd function	$x \neq 0$
$\operatorname{sech}^{-1} x$	$0 < x \leq +1$	$0 \leq y < +\pi$		
$\coth^{-1} x$	$x < -1 \quad +1 < x$	$y \in \mathbb{P}$	odd function	

Notes

[†] sech^2 is a bell shaped distribution occurring in the natural world.

First Order Chebyshev Polynomials

$$\begin{aligned}
 T_0(x) &= 1 \\
 T_1(x) &= x \\
 T_2(x) &= 2x^2 - 1 \\
 T_3(x) &= 4x^3 - 3x \\
 T_4(x) &= 8x^4 - 8x^2 + 1 \\
 T_5(x) &= 16x^5 - 20x^3 + 5x \\
 T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1 \\
 T_7(x) &= 64x^7 - 112x^5 + 56x^3 - 7x \\
 T_8(x) &= 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1
 \end{aligned}$$

$$\begin{aligned}
 T_0(x) &= 1 \\
 T_1(x) &= x \\
 T_{n+1}(x) &= 2x T_n(x) - T_{n-1}(x) \\
 \text{whence } T_n(\cos \theta) &= \cos(n\theta)
 \end{aligned}$$

Second Order Chebyshev Polynomials

$$\begin{aligned}
 U_0(x) &= 1 \\
 U_1(x) &= 2x \\
 U_2(x) &= 4x^2 - 1 \\
 U_3(x) &= 8x^3 - 4x \\
 U_4(x) &= 16x^4 - 12x^2 + 1 \\
 U_5(x) &= 32x^5 - 32x^3 + 6x \\
 U_6(x) &= 64x^6 - 80x^4 + 24x^2 - 1 \\
 U_7(x) &= 128x^7 - 192x^5 + 80x^3 - 8x \\
 U_8(x) &= 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1
 \end{aligned}$$

$$\begin{aligned}
 U_0(x) &= 1 \\
 U_1(x) &= 2x \\
 U_{n+1}(x) &= 2x U_n(x) - U_{n-1}(x) \\
 \text{whence } U_n(\cos \theta) &= \frac{\sin(n+1)\theta}{\sin \theta}
 \end{aligned}$$

Spread Polynomials

$$\begin{aligned}
 S_0(s) &= 0 \\
 S_1(s) &= s \\
 S_2(s) &= -4s^2 + 4s \\
 S_3(s) &= 16s^3 - 24s^2 + 9s \\
 S_4(s) &= -64s^4 + 128s^3 - 80s^2 + 16s \\
 S_5(s) &= 256s^5 - 640s^4 + 560s^3 - 200s^2 + 25s \\
 S_6(s) &= -1024s^6 + 3072s^5 - 3456s^4 + 1792s^3 - 420s^2 + 36s \\
 S_7(s) &= 4096s^7 - 14336s^6 + 19712s^5 - 13440s^4 + 4704s^3 - 784s^2 + 49s \\
 S_8(s) &= -16384s^8 + 65536s^7 - 106496s^6 + 90112s^5 - 42240s^4 + 10752s^3 - 1344s^2 + 64s
 \end{aligned}$$

$$\sin^2(n\theta) = S_n(\sin^2\theta)$$

Counting

No.	Greek	Latin
1	mono	uni
2	duo	bi
3	tri	tri
4	tetra	quad
5	penta	quin
6	hexa	sex
7	hepta	sept
8	octo	oct
9	nona	non
10	deca	dec

The "Handbook of Mathematics" has been written and produced by Robert Goodhand

Although the formulae and expressions given have been individually derived and checked errors do creep in. Sections of the Handbook are continuously updated.

If you would like the latest issue, just email me at robert.goodhand@gmail.com