An Investigation into sin 18° and the golden ratio

sin 18°

let $\varphi = 18°$

then $5\varphi = 90°$

$5\varphi \quad = 90° - 3\varphi$

$\sin 2\varphi = \sin (90° - 3\varphi)$

$\sin 2\varphi = \cos 3\varphi$

$2\sin \varphi \cos\varphi$

$$= 4 \cos^3\varphi - 3\cos\varphi$$

$2\sin \varphi \qquad = 4 \cos^2\varphi - 3$

$2\sin \varphi = 4 (1 - \sin^2\varphi) - 3$

$4\sin^2 \varphi + 2 \sin\varphi - 1 = 0$

$2\sin^2 \varphi + \sin\varphi - ½ = 0$

$\sin\varphi = - 1 \pm \sqrt{\{1-(4)(2)(-½)\}}$

which all simplifies down to

$\sin\varphi \qquad = ¼ (\sqrt{5}-1)$

so $2 \sin\varphi$ = golden ratio

cos 18°

$\cos^2\varphi \qquad = 1 - \sin^2\varphi$

$\cos \varphi \qquad = \sqrt{(1 - \sin^2\varphi)}$

$= \sqrt{\{ 1 - {}^1/_{16} (6 - 2\sqrt{5}) \}}$

$= \sqrt{{}^1/_{16}(10 + 2\sqrt{5})}$

$= \sqrt{{}^1/_8 (5 + \sqrt{5})}$

sin 36°

$\sin 2\varphi = 2 \sin \varphi \cos\varphi$

$= 2 (¼ (\sqrt{5} - 1)) \{ \sqrt{{}^1/_8 (5 + \sqrt{5})} \}$

which if you bother to work out is

$$= ¼ \sqrt{(10 - 2\sqrt{5})}$$

cos 36°(2φ)

The easiest expression to work with is

$\cos 2\varphi \qquad = 1 - 2 \sin^2 \varphi$

$$= 1 - 2 (¼)^2(\sqrt{5} -1)^2$$

which simplifies to

$$¼ (1 + \sqrt{5})$$

sin 72°

again $\sin 4\varphi$

$$= 2 \sin 2\varphi \cos 2\varphi$$

=

$2(\frac{1}{4})\frac{1}{4}\sqrt{(10-2\sqrt{5})}\{\frac{1}{4}(1\sqrt{5})\}$

which simplifies to

$$\tfrac{1}{4}\sqrt{(10 + 2\sqrt{5})}$$

cos 72º

$$\text{try } \cos^2\varphi = 1 - \sin^2\varphi$$

$$= 1 - \tfrac{1}{16}(10 + 2\sqrt{5})$$

$$\text{so } \cos\varphi = \tfrac{1}{4}\sqrt{(6 - 2\sqrt{5})}$$

$$= \tfrac{1}{4}\sqrt{\{(\sqrt{5} - 1)^2\}}$$

$$= \tfrac{1}{4}(\sqrt{5} - 1)$$

Golden Ratio

$$^{\sin 36^\circ}/_{\sin 72^\circ} = {}^{\frac{1}{4}\sqrt{(10 - 2\sqrt{5})}} / {}_{\frac{1}{4}\sqrt{(10 + 2\sqrt{5})}}$$

$$= \sqrt{\{^{(120 - 40\sqrt{5})}/_{80}\}}$$

$$= \sqrt{\{^{(3 - \sqrt{5})}/_{2}\}}$$

$$= \tfrac{1}{2}\sqrt{\{(6 - 2\sqrt{5})^2\}}$$

$$= \tfrac{1}{2}(\sqrt{5} - 1)$$

(see Appendix)

which is the golden ratio.

If you draw a pentagon and connect the vertices the golden ratio crops up repeatedly. And this is why.

Appendix

That $(\sqrt{5} - 1)^2 = 6 - 2\sqrt{5}$ is interesting and I would have missed it if I didn't happen to know where I was heading and needed to end up.

The sequence continues

$(\sqrt{5} - 2)^2 = 9 - 4\sqrt{5}$

$(\sqrt{5} - 3)^2 = 14 - 6\sqrt{5}$

$(\sqrt{5} - 4)^2 = 21 - 8\sqrt{5}$

but none of these occur in the previous calculations so nothing missed.

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$2\sin \varphi = 4 (1 - \sin^2\varphi) - 3$

$4\sin^2 \varphi + 2 \sin\varphi - 1 = 0$

$2\sin^2 \varphi + \sin\varphi - \tfrac{1}{2} = 0$

$\sin\varphi = - 1 \pm \sqrt{\{1 - (4)(2)(-\tfrac{1}{2})}$

$\}$

which all simplifies down to

$\sin\varphi \qquad = \tfrac{1}{4} (\sqrt{5}-1)$

so $2 \sin\varphi$ = golden ratio

cos 18°

$\cos^2\varphi \qquad = 1 - \sin^2\varphi$

$\cos \varphi \qquad = \sqrt{ (1 - \sin^2\varphi)}$

$= \sqrt{ \{ 1 - {}^1/_{16} (6 - 2\sqrt{5}) \} }$

$$= \sqrt{ {}^1/_{16}(10 + 2\sqrt{5})}$$

$$= \sqrt{ {}^1/_8 (5 + \sqrt{5})}$$

sin 36°

$\sin 2\varphi = 2 \sin \varphi \cos\varphi$

$= 2 (\tfrac{1}{4} (\sqrt{5} - 1)) \{ \sqrt{ {}^1/_8 (5 +}$

$\sqrt{5}) \}$

which if you bother to work

out is

$$= \tfrac{1}{4} \sqrt{(10 - 2\sqrt{5})}$$

cos 36°(2φ)

The easiest expression to

work with is

$\cos 2\varphi \qquad = 1 - 2 \sin^2 \varphi$

$$= 1 - 2 (\tfrac{1}{4})^2(\sqrt{5} -1)^2$$

which simplifies to

$$\tfrac{1}{4} (1 + \sqrt{5})$$

sin 72°

again $\sin 4\varphi$

$$= 2 \sin 2\varphi \cos 2\varphi$$

=

$2(\tfrac{1}{4})\tfrac{1}{4}\sqrt{(\ 10-2\sqrt{5}\ )}\{\tfrac{1}{4}(1\ \ \sqrt{5}\ )\}$

which simplifies to

$$\tfrac{1}{4}\sqrt{(\ 10+2\sqrt{5}\ )}$$

cos 72°

try $\cos^2\varphi = 1 - \sin^2\varphi$

$= 1 - {}^{1}/_{16}(10 + 2\sqrt{5})$

so $\cos\varphi = \tfrac{1}{4}\sqrt{(\ 6 - 2\sqrt{5}\ )}$

$= \tfrac{1}{4}\sqrt{\{\ (\ \sqrt{5} - 1\ )^2\}}$

$= \tfrac{1}{4}(\ \sqrt{5} - 1\ )$

Golden Ratio

$${}^{\sin 36°}/_{\sin 72°} = {}^{\frac{1}{4}\sqrt{(\ 10-2\sqrt{5}\ )}}/_{\frac{1}{4}\sqrt{(\ 10+2\sqrt{5}\ )}}$$

$$= \sqrt{\{\ {}^{(\ 120 - 40\sqrt{5}\ )}/_{80}\}}$$

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$$= \tfrac{1}{2}\sqrt{\{\ (6 - 2\sqrt{5}\ )^2\ \}}$$

$$= \tfrac{1}{2}(\sqrt{5} - 1\ )$$

(see Appendix)

which is the golden ratio.

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