Periodicity Trig Functions Iss 2

Introduction

$e^x = \cosh x + \sinh x$

That is cosh x and sinh x are respectively the even part and odd part of the exponential function.

Now let $x \to ix$ and hence

$e^{ix} = \cosh ix + \sinh ix$

but we already have by Euler

$e^{ix} = \cos x + i \sin x$ so

$\cosh ix = \cos x$ and $\sinh ix = i \sin x$

Now let $x \to {}^x/_i$ and hence

$\cosh x \quad = \cos {}^x/_i$

$\quad = \cos(-ix)$

and given that cos is an even function and holds in the complex plane $\cosh x \quad = \cos ix$ and

$\sinh x = i \sin {}^x/_i = i \sin(-ix)$

and given that sin is an odd function and holds in the complex plane

$\sinh x \quad = -i \sin ix$

Hyperbolic Periodicity

$\sinh(x + \tfrac{1}{2}i\pi)$

$= \sinh x \cosh \tfrac{1}{2}i\pi + \cosh x \sinh \tfrac{1}{2}i\pi$

$\cosh \tfrac{1}{2}i\pi = \cos \tfrac{1}{2}\pi = 0$

$\sinh \tfrac{1}{2}i\pi = i \sin\tfrac{1}{2}\pi = i$

so $\sinh(x + \tfrac{1}{2}i\pi) = i \cosh x$

$\sinh(x + i\pi)$

$= \sinh x \cosh i\pi + \cosh x \sinh i\pi$

$\cosh i\pi = \cos \pi = -1$

$\sinh i\pi = i \sin\pi = 0$

so $\sinh(x + i\pi) = -\sinh x$

$\sinh(x + {}^3/_2 i\pi)$

$= \sinh x \cosh {}^3/_2 i\pi + \cosh x \sinh {}^3/_2 i\pi$

$\cosh {}^3/_2 i\pi = \cos {}^3/_2 \pi = 0$

$\sinh {}^3/_2 i\pi = i \sin {}^3/_2 \pi = -i$

so $\sinh(x + {}^3/i\pi) = -i \cosh x$

$\sinh(x + 2i\pi)$

$= \sinh x \cosh 2i\pi + \cosh x \sinh 2i\pi$

$\cosh 2i\pi = \cos 2\pi = 1$

$\sinh 2i\pi = i \sin\pi = 0$

so $\sinh(x + 2i\pi) = \sinh x$

$\cosh(x + \tfrac{1}{2}i\pi)$

$= \text{coshx } x \cosh \tfrac{1}{2}i\pi + \sinh x \sinh \tfrac{1}{2}i\pi$

$\cosh \tfrac{1}{2}i\pi = \cos \tfrac{1}{2}\pi = 0$

$\text{sinhx } \tfrac{1}{2}i\pi = i \sin\tfrac{1}{2}\pi = i$

so $\text{coshx}(x + \tfrac{1}{2}i\pi) = i \sinh x$

$\cosh(x + i\pi)$

$= \text{coshx } x \cosh i\pi + \sinh x \text{ sinhx } i\pi$

$\cosh i\pi = \cos \pi = -1$

$\sinh i\pi = i \sin\pi = 0$

so $\cosh(x + i\pi) = -\cosh x$

$\text{coshx}(x + {}^3/_2 i\pi)$

$= \text{coshx } x \cosh {}^3/_2 i\pi +$

$\sinh x \text{ sinhx } {}^3/_2 i\pi$

$\cosh {}^{3}/_{2}i\pi = \cos {}^{3}/_{2}\pi = 0$

$\sinh {}^{3}/_{2}i\pi = i \sin {}^{3}/_{2}\pi = -i$

so $\cosh (x + {}^{3}/i\pi) = - i \sinh x$

$\cosh (x + 2i\pi)$

$= \cosh x \sinh 2i\pi + \sinh x \sinh 2i\pi$

$\cosh 2i\pi = \cos 2\pi = 1$

$\sinh 2i\pi = i \sin\pi = 0$

so $\cosh (x + 2i\pi) = \cosh x$

So for sinh and cosh each ½iπ advance in phase cosh → sinh and sinh → cosh and the factor i is introduced. So after 4 × ½iπ the cycle completes.

Circular Periodicity

$\sin (x + \frac{1}{2}\pi)$

$= \sin x \cos \frac{1}{2}\pi + \cos x \sin \frac{1}{2}\pi$

$\cos \frac{1}{2}\pi = 0 \qquad \sin \frac{1}{2}\pi = 1$

so $\sin (x + \frac{1}{2}\pi) = \cos x$

$\sin (x + \pi)$

$= \sin x \cos \pi + \cos x \sin \pi$

$\cos \pi = -1 \qquad \sin \pi = 0$

so $\sin (x + \pi) = - \sin x$

$\sin (x + {}^{3}/_{2}\pi)$ AA

$= \sin x \cos {}^{3}/_{2}\pi + \cos x \sin {}^{3}/_{2}\pi$

$\cos {}^{3}/_{2}\pi = 0 \qquad \sin {}^{3}/_{2}i\pi = -1$

so $\sin (x + {}^{3}/_{2}\pi) = - \cos x$

$\sin (x + 2\pi)$

$= \sin x \cos 2\pi + \cos x \sin 2\pi$

$\cos 2\pi = 1 \qquad \sin 2\pi = 0$

so $\sin (x + 2\pi) = \sin x$

$\cos (x + \frac{1}{2}\pi)$

$= \cos x \cos \frac{1}{2}\pi - \sin x \sin \frac{1}{2}\pi$

$\cos \frac{1}{2}\pi = 0 \qquad \sin x \frac{1}{2}\pi = 1$

so $\cos x (x + \frac{1}{2}\pi) = - \sin x$

$\cos (x + \pi)$

$= \cos x \cos \pi - \sin x \sin \pi$

$\cos \pi = -1 \qquad \sin \pi = 0$

so $\cos (x + \pi) = - \cos x$

$\cos x (x + {}^{3}/_{2}i\pi)$

$= \cos x \cos {}^{3}/_{2}\pi - \sin x \sin x {}^{3}/_{2}\pi$

$\cos {}^{3}/_{2}\pi = 0 \qquad \sin {}^{3}/_{2}\pi = -1$

so $\cos (x + {}^{3}/_{2}\pi) = \sin x$

$\cos (x + 2\pi)$

$= \cos x \cos 2\pi - \sin x \sin 2\pi$

$\cos 2\pi = 1 \qquad \sin 2\pi = 0$

so $\cos (x + 2\pi) = \cos x$

So sin and cos display a certain asymmetry. For each ½π advance in phase cos → –sin but sin → cos. So after 4 × ½π the cycle completes.

Conclusion

Both sin/cos and sinh/cosh have a 2π / 2iπ periodicity but for slightly different reasons *rg*

Introduction

$e^x = \cosh x + \sinh x$

That is cosh x and sinh x are respectively the even part and odd part of the exponential function.

Now let $x \to ix$ and hence

$e^{ix} = \cosh ix + \sinh ix$

but we already have by Euler

$e^{ix} = \cos x + i \sin x$ so

$\cosh ix = \cos x$ and $\sinh ix = i \sin x$

Now let $x \to {}^x/_i$ and hence

$\cosh x = \cos {}^x/_i$

$= \cos(-ix)$

and given that cos is an even function and holds in the complex plane $\cosh x = \cos ix$ and

$\sinh x = i \sin {}^x/_i = i \sin(-ix)$

and given that sin is an odd function and holds in the complex plane

$\sinh x = -i \sin ix$

Hyperbolic Periodicity

$\sinh(x + \tfrac{1}{2}i\pi)$

$= \sinh x \cosh \tfrac{1}{2}i\pi + \cosh x \sinh \tfrac{1}{2}i\pi$

$\cosh \tfrac{1}{2}i\pi = \cos \tfrac{1}{2}\pi = 0$

$\sinh \tfrac{1}{2}i\pi = i \sin\tfrac{1}{2}\pi = i$

so $\sinh(x + \tfrac{1}{2}i\pi) = i \cosh x$

$\sinh(x + i\pi)$

$= \sinh x \cosh i\pi + \cosh x \sinh i\pi$

$\cosh i\pi = \cos \pi = -1$

$\sinh i\pi = i \sin\pi = 0$

so $\sinh(x + i\pi) = -\sinh x$

$\sinh(x + {}^3/_2 i\pi)$

$= \sinh x \cosh {}^3/_2 i\pi + \cosh x \sinh {}^3/_2 i\pi$

$\cosh {}^3/_2 i\pi = \cos {}^3/_2 \pi = 0$

$\sinh {}^3/_2 i\pi = i \sin {}^3/_2 \pi = -i$

so $\sinh(x + {}^3/i\pi) = -i \cosh x$

$\sinh(x + 2i\pi)$

$= \sinh x \cosh 2i\pi + \cosh x \sinh 2i\pi$

$\cosh 2i\pi = \cos 2\pi = 1$

$\sinh 2i\pi = i \sin\pi = 0$

so $\sinh(x + 2i\pi) = \sinh x$

$\cosh(x + \tfrac{1}{2}i\pi)$

$= \cosh x \cosh \tfrac{1}{2}i\pi + \sinh x \sinh \tfrac{1}{2}i\pi$

$\cosh \tfrac{1}{2}i\pi = \cos \tfrac{1}{2}\pi = 0$

$\sinh x\, \tfrac{1}{2}i\pi = i \sin\tfrac{1}{2}\pi = i$

so $\cosh x(x + \tfrac{1}{2}i\pi) = i \sinh x$

$\cosh(x + i\pi)$

$= \cosh x \cosh i\pi + \sinh x \sinh x\, i\pi$

$\cosh i\pi = \cos \pi = -1$

$\sinh i\pi = i \sin\pi = 0$

so $\cosh(x + i\pi) = -\cosh x$

$\cosh x(x + {}^3/_2 i\pi)$

$= \cosh x \cosh {}^3/_2 i\pi + \sinh x \sinh x\, {}^3/_2 i\pi$

$\cosh {}^{3}/_{2}i\pi = \cos {}^{3}/_{2}\pi = 0$

$\sinh {}^{3}/_{2}i\pi = i \sin {}^{3}/_{2}\pi = -i$

so $\cosh (x + {}^{3}/i\pi) = -\, i \sinh x$

$\cosh (x + 2i\pi)$

$= \cosh x \sinh 2i\pi + \sinh x \sinh 2i\pi$

$\cosh 2i\pi = \cos 2\pi = 1$

$\sinh 2i\pi = i \sin\pi = 0$

so $\cosh (x + 2i\pi) = \cosh x$

So for sinh and cosh each $\frac{1}{2}i\pi$ advance in phase cosh $\rightarrow$ sinh and sinh $\rightarrow$ cosh and the factor i is introduced. So after $4 \times \frac{1}{2}i\pi$ the cycle completes.

Circular Periodicity

$\sin (x + \frac{1}{2}\pi)$

$= \sin x \cos \frac{1}{2}\pi + \cos x \sin \frac{1}{2}\pi$

$\cos \frac{1}{2}\pi = 0 \qquad \sin \frac{1}{2}\pi = 1$

so $\sin (x + \frac{1}{2}\pi) = \cos x$

$\sin (x + \pi)$

$= \sin x \cos \pi + \cos x \sin \pi$

$\cos \pi = -1 \qquad \sin \pi = 0$

so $\sin (x + \pi) = -\sin x$

$\sin (x + {}^{3}/_{2}\pi)$ AA

$= \sin x \cos {}^{3}/_{2}\pi + \cos x \sin {}^{3}/_{2}\pi$

$\cos {}^{3}/_{2}\pi = 0 \qquad \sin {}^{3}/_{2}i\pi = -1$

so $\sin (x + {}^{3}/_{2}\pi) = -\cos x$

$\sin (x + 2\pi)$

$= \sin x \cos 2\pi + \cos x \sin 2\pi$

$\cos 2\pi = 1 \qquad \sin 2\pi = 0$

so $\sin (x + 2\pi) = \sin x$

$\cos (x + \frac{1}{2}\pi)$

$= \cos x \cos \frac{1}{2}\pi - \sin x \sin \frac{1}{2}\pi$

$\cos \frac{1}{2}\pi = 0 \qquad \text{sinx } \frac{1}{2}\pi = 1$

so $\text{cosx } (x + \frac{1}{2}\pi) = -\sin x$

$\cos (x + \pi)$

$= \cos x \cos \pi - \sin x \sin \pi$

$\cos \pi = -1 \qquad \sin \pi = 0$

so $\cos (x + \pi) = -\cos x$

$\text{cosx } (x + {}^{3}/_{2}i\pi)$

$= \cos x \cos {}^{3}/_{2}\pi - \sin x \text{ sinx } {}^{3}/_{2}\pi$

$\cos {}^{3}/_{2}\pi = 0 \qquad \sin {}^{3}/_{2}\pi = -1$

so $\cos (x + {}^{3}/_{2}\pi) = \sin x$

$\cos (x + 2\pi)$

$= \cos x \cos 2\pi - \sin x \sin 2\pi$

$\cos 2\pi = 1 \qquad \sin 2\pi = 0$

so $\cos (x + 2\pi) = \cos x$

So sin and cos display a certain asymmetry. For each $\frac{1}{2}\pi$ advance in phase cos $\rightarrow$ $-$sin but sin $\rightarrow$ cos. So after $4 \times \frac{1}{2}\pi$ the cycle completes.

Conclusion

Both sin/cos and sinh/cosh have a 2π / $2i\pi$ periodicity but for slightly different reasons *rg*