An Investigation into Cubic Symmetry

Any cubic equation has rotational symmetry about the point of inflexion. That point can be translated to the x-y coordinate intersect to give an "odd" function.

This leaflet investigates an arbitrary case and then uses that as the template for a general solution.

The idea for this came from a YouTube video by Dr Peyam but pursues an alternative proof.

Specific Solution

Take the function

$f\,(x) = x^3 - 3x^2 + {}^9/_4\, x + 6$

$f\,'(x) = 3x^2 - 6x + {}^9/_4$

which we could solve by the general formula but the coefficients have been chosen for factorising

$f\,'(x) = (\,3x - {}^3/_2\,)\,(\,x - {}^1/_2\,)$

hence $x_1 = ½$ and $x_2 = 1½$

and $y_1 = 6$ and $y_2 = 6½$

Continuing $f\,''(x) = 6x^2 - 6$

$x_3 = 1$ and $y_3 = 6¼$

It is immediately apparent that

x_3 (1) lies equidistant between

x_1 (½) and x_2 (1½) and

y_3 (6 ¼)lies equidistant between

y_1 (6) and y_2 (6 ½)

which although not a formal proof a clear indication that the function has rotation symmetry about (x_3 , y_3)

The next step is to translate f (x) so the point of symmetry lies over the coordinate intersect.

$f\,(x) = x^3 - 3x^2 + {}^9/_4\, x + 6$

$x \Rightarrow (\,x + 1)$ and $y \Rightarrow y - 6\,¼$

$g\,(x) = (\,x^3 + 1\,)^3 - (\,x + 1)^{\,2}$

$\qquad + {}^9/_4\,(\,x + 1\,) + 6 - 6\,¼$

$g\,(x) = x^3 + 3x^2 + 3x + 1$

$\qquad - 3\,(\,x^2 + 2x + 1\,)$

$\qquad + {}^9/_4\,(\,x + 1\,) - 6\,¼$

and we see the even terms x^2 and x^o conveniently cancel out leaving

$g\,(x) = x^3 - ¾\,x$

which is expected but still a miracle.

To summarise we have moved a typical cubic equation so that the point of inflection lines up with coordinate (0, 0). When we substitute those x and y translations into the original equation we get a new function composed only of "odd" terms – that is an "odd" function which by definition has rotational symmetry.

General Solution

Take the function

$f(x) = ax^3 + bx^2 + cx + d$

$f'(x) = 3ax^2 + 2bx + c$

$f''(x) = 6ax + 2b$

Hence at point of inflexion $x = -{}^{b}/_{3a}$

$f(-{}^{b}/_{3a}) = a(-{}^{b}/_{3a})^3 + b(-{}^{b}/_{3a})^2 + c(-{}^{b}/_{3a}) + d$

which by manipulation will produce

$f(-{}^{b}/_{3a}) = -{}^{b}/_{3a}\,({}^{b^2}/_{9a^2} - {}^{b^2}/_{3a^2} + c) + d$

Now referring to the specific example we have

$a = 1$ $b = -3$ $c = {}^{9}/_{4}$ and $d = 6$

Hence $-{}^{b}/_{3a} = 1$ which is x_3 and

$-{}^{b}/_{3a}\,({}^{b^2}/_{9a} - {}^{b^2}/_{3a} + c) + d = 6\tfrac{1}{4}$ (y_3)

Embarking on a feast of algebra

$x \Rightarrow (x - {}^{b}/_{3a})$ and

$y \Rightarrow y + {}^{b^3}/_{27a^2} - {}^{b^3}/_{9a^2} + {}^{bc}/_{3a} - d$

$$g(x) = a(x^3 - {}^{b}/_{3a})^3 + b(x - {}^{b}/_{3a})^2 + c(x - {}^{b}/_{3a}) + d + {}^{b^3}/_{27a^2} - {}^{b^3}/_{9a^2} + {}^{cb}/_{3a} - d$$

and by binomial expansion $g(x) =$

$$a\{x^3 + 3x^2(-{}^{b}/_{3a}) + 3x(-{}^{b}/_{3a})^2 + (-{}^{b}/_{3a})^3\} + b\{x^2 + 2x(-{}^{b}/_{3a}) + (-{}^{b}/_{3a})^2\} + c\{x^2 - {}^{b}/_{3a}) + {}^{b^3}/_{27a^2} - {}^{b^3}/_{9a^2} + {}^{bc}/_{3a}$$

where we have eliminated d at the outset.

Multiplying out and gathering terms

$g(x) = ax^3 + {}^{b^2x}/_{3a} - {}^{2b^2x}/_{3a} + cx$

$g(x) = ax^3 + (c - {}^{b^2}/_{3a})\,x$

So if we go back again to the specific example $a = 1$ $b = -3$ $c = {}^{9}/_{4}$ and $d = 6$

$(c - {}^{b^2}/_{3a}) = -\tfrac{3}{4}$ as expected

To find the slope at the point of inflexion we substitute into $f'(x)$ the value $-{}^{b}/_{3a}$

$$f'(-{}^{b}/_{3a}) = 3a(-{}^{b}/_{3a})^2 + 2b(-{}^{b}/_{3a}) + c = c - {}^{b^2}/_{3a}$$

The term $(c - {}^{b^2}/_{3a})$ recurs which means we can say

$g(x) = ax^3 + f'(-{}^{b}/_{3a})\,x$

RG claims this as an internet first

Summary

$F(x) = ax^3 + bx^2 + cx + d$

Rotational symmetry about is about $x = -{}^{b}/_{3a}$ and can be transformed to the odd function

$g(x) = ax^3 + (c - {}^{b^2}/_{3a})\,x$ or

$g(x) = ax^3 + f'(-{}^{b}/_{3a})\,x$

XORg

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which by manipulation will produce

$f(-{}^{b}/_{3a}) = -{}^{b}/_{3a}\,(\,{}^{b^2}/_{9a^2} - {}^{b^2}/_{3a^2} + c\,) + d$

Now referring to the specific example we have

$a = 1\ b = -3\ c = {}^{9}/_{4}$ and $d = 6$

Hence $-{}^{b}/_{3a} = 1$ which is x_3 and

$-{}^{b}/_{3a}\,(\,{}^{b^2}/_{9a} - {}^{b^2}/_{3a} + c\,) + d = 6\ ¼\ \ (y_3)$

Embarking on a feast of algebra

$x \Rightarrow (\,x - {}^{b}/_{3a}\,)$ and

$y \Rightarrow y + {}^{b^3}/_{27a^2} - {}^{b^3}/_{9a^2} + {}^{bc}/_{3a} - d$

$g(x) = a\,(\,x^3 - {}^{b}/_{3a}\,)^3 + b(\,x - {}^{b}/_{3a}\,)^2$
$+ c\,(\,x - {}^{b}/_{3a}\,) + d$
$+ {}^{b^3}/_{27a^2} - {}^{b^3}/_{9a^2} + {}^{cb}/_{3a} - d$

and by binomial expansion $g(x) =$

$a\{x^3+3x^2(-{}^{b}/_{3a})+3x(-{}^{b}/_{3a})^2+(-{}^{b}/_{3a})^3\}$
$+b\{x^2+2x(-{}^{b}/_{3a})+(-{}^{b}/_{3a})^2\}$
$+c\{x^2-{}^{b}/_{3a})+{}^{b^3}/_{27a^2} - {}^{b^3}/_{9a^2} + {}^{bc}/_{3a}$

where d eliminated at the outset.

Multiplying out and gathering terms

$g(x) = ax^3 + {}^{b^2x}/_{3a} - {}^{2b^2x}/_{3a} + cx$

$g(x) = ax^3 + (\,c - {}^{b^2}/_{3a}\,)\,x$

So if we go back again to the specific example $a = 1\ b = -3\ c = {}^{9}/_{4}$ and $d = 6$

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To find the slope at the point of inflexion we substitute into $f'(x)$ the value $-\,{}^{b}/_{3a}$

$f'(-\,{}^{b}/_{3a}) = 3a\,(-\,{}^{b}/_{3a})^2 + 2b\,(-\,{}^{b}/_{3a})+c$
$= c - {}^{b^2}/_{3a}$

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$g(x) = ax^3 + f'(-\,{}^{b}/_{3a})\,x$

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Summary

$F(x) = ax^3 + bx^2 + cx + d$

Rotational symmetry about is about $x = -\,{}^{b}/_{3a}$ transformed to the odd function $g(x) = ax^3 + (\,c - {}^{b^2}/_{3a}\,)\,x$

or $g(x) = ax^3 + f'(-\,{}^{b}/_{3a})\,x$

Rg