Mr G's Little Book on

Maxwell's Equations

with particular emphasis on dimensional analysis

Introduction

Maxwell's equations have equal importance in defining the underlying structure of the Universe as Newton's Laws of Motion or Einstein's equations of Special and General Relativity.

This is a revision guide to those equations including a thorough review of all electrostatic and magnetostatic units and terms. It reviews the underlying mathematics and logical conclusions of each of the four equations.

James Clerk Maxwell published a two volume "Treatise on Electricity and Magnetism" in 1873. It contained 20 key equations. Maxwell derived these using quaternions, a precursor of vector cross multiplication.

Oliver Heaviside distilled these into four equations by applying the new principles of dot and cross multiplication in vector algebra.

This is a good example of Stigler's law of eponymy, proposed by University of Chicago statistics professor Stephen Stigler. It states that no scientific discovery is named after its original discoverer. Stigler's Law is however an exception to this rule.

Rationalised Units

The author completed his Physics "A" Levels using the cgs system and started his Engineering degree using the MKS system. The tension between mathematical simplicity and practical units is apparent. Even the cgs system is actually two systems – the cgs –esu and the cgs – emu. A table in this booklet gives the conversion between the three systems and presents a bewildering array of units. However the author noticed even during his "A" levels that the ratio of the basic electrical units of current, charge and potential difference in esu units and emu units gave the numerical value of c in cgs units, the first indication that light is an electromagnetic phenomenon.

Vector Revision

A vector **v** has magnitude and direction

Let **i** **j** **k** represent unit vectors along coordinate axii x y and z

$\underline{\mathbf{v}} = v_1\underline{\mathbf{i}} + v_2\underline{\mathbf{j}} + v_3\underline{\mathbf{k}}$

Magnitude (mod A)

$|\underline{\mathbf{v}}| = \sqrt{(v_1^2 + v_2^2 + v_3^2)}$

Scalar dot Product •

$\underline{\mathbf{v}} \bullet \underline{\mathbf{w}} = |v|\ |w| \cos\theta$

$\underline{\mathbf{v}} \bullet \underline{\mathbf{w}} = v_1w_1 + v_2w_2 + v_3w_3$

The dot product gives the magnitude of the force vector applied in the direction of the motion vector.

Vector Cross Product ×

$\underline{\mathbf{v}} \times \underline{\mathbf{w}} = |v|\ |w| \sin\theta\ \underline{\mathbf{u}}$

The cross product is defined as a vector that is perpendicular to both v and w, with a direction given by the right-hand rule and a magnitude equal to the area of the parallelogram that the vectors span.

A vector cross product is only defined in 2^n-1 dimensions. Cross multiplication in 3 dimensions relates to quaternions and in 7 dimensions to octonions.

Concept of a Field

An **electric field** (sometimes **E-field**) is the physical field that surrounds electrically charged particles and exerts force on all other charged particles in the field, either attracting or repelling them. An electric line of force is the path followed by a free electrically charged particle in an electric field.

A **magnetic field** is a vector field that describes the magnetic influence on moving electric charges, electric currents and magnetic materials. A moving charge in a magnetic field experiences a force perpendicular to its own velocity and to the magnetic field.

Del, Grad Div and Curl

Del

The vector operator ∇ pronounced "del" is defined as

$\nabla = i\ {}^{\partial}/_{\partial x} + j\ {}^{\partial}/_{\partial x} + k\ {}^{\partial}/_{\partial x}$

Grad

∇f is pronounced "grad f" and indicates the partial differential to be carried out on the scalar function f.

The gradient of a scalar function is a vector with the following properties

1) Its components at any point are the

rates of change of the function along the directions of the coordinate axes at that point.

2) Its magnitude at the point is the maximum rate of change of the function with distance

3) Its direction is that of the maximum rate of change of the function and it points towards larger values of the function.

Div

Next we define **A** as a vector point function – ie a function that defines a vector at each point in space – then

div **A** = $\nabla \cdot \mathbf{A}$ and pronounced "div A".

This is the scalar dot product between a vector and a vector operator.

It is a local measure of its "outgoingness" – the extent to which there are more of the field vectors exiting an infinitesimal region of space than entering it. Positive outgoing flux is called a source and negative divergence a sink. The greater the flux of field, the greater the value of divergence at that point. A point at which there is zero flux through an enclosing surface has zero divergence. In fluid dynamics, a flow is considered incompressible if the divergence of the flow velocity is zero.

Curl

Finally curl **A** = $\nabla \times \mathbf{A}$ pronounced "curl A" is the vector cross product between a vector operator and a vector. Vector cross products are only defined for 3 (and 7) dimensions. It is a vector that describes the infinitesimal circulation of a vector field in 3-D Euclidean space.

Finally the divergence of the curl of any vector field is always 0.

$\nabla \cdot (\nabla \times \mathbf{A}) = 0$

Summary

∇ is a vector operator

∇f is a vector quantity giving the gradient of a scalar point function.

$\nabla \cdot \mathbf{A}$ is a scalar quantity giving the divergence of a region of space.

$\nabla \times \mathbf{A}$ is a vector quantity giving the rotation of a region of space.

Concept of Flux

Flux whether electric or magnetic is the measure of the effective field – the vector lines of force – passing at right angles through a given surface area.

Dimensional Analysis

Force

Force = mass × acceleration [$M\ L\ T^{-2}$] and is measured in *newtons.*

Energy

Energy is the potential to do work where work = force × distance [$M\ L^2\ T^{-2}$] and is measured in *joules.*

Charge

Charge is denoted Q. It is a fundamental unit measured in *coulombs.*

Current

In a conductive medium the rate of flow is measured in *amperes* i [$Q\ T^{-1}$].

Current Density J

The current density **J** is a vector measured in $^{\text{amperes}}/_{\text{metre}^2}$ [$QT^{-1}L^{-2}$]. The convection current is denoted $\mathbf{J_c}$, any dielectric polarisation current $\mathbf{J_p}$ and the equivalent magnetic current $\mathbf{J_M}$.

Vector Operators

Both $\nabla\bullet$ and $\nabla\times$ have in this context the dimension L^{-1}.

Electrostatics

1) Electric Flux Φ_E

Electric flux Φ_E, a vector quantity, is defined as $^{\text{force}}/_{\text{charge density}}$ and hence measured in $^{\textit{newton-metre}^2}/_{\textit{coulomb}}$. The dimensions are [$ML^3\ Q^{-1}T^{-2}$] or *volts-metre*. This will be derived later.

2) Charge density D D= $^{dQ}/_{dA}$

The charge density **D** – also termed the electric displacement – is $^{\text{charge}}/_{\text{area}}$ $^{\textit{coulombs}}/_{\textit{metre}^2}$ [$Q\ L^{-2}$].

This may also be referred to as the surface charge density σ. This should be distinguished from the linear charge density λ $^{\textit{coulombs}}/_{\textit{metre}}$ [$Q\ L^{-1}$] and volume charge density ρ $^{\textit{coulombs}}/_{\textit{metre}^3}$ [$Q\ L^{-3}$]

3) Electro-motive Force emf

Energy per charge ie $^{\textit{joules}}/_{\textit{coulomb}}$ is the electrical potential difference measured in *volts* [$M\ L^2\ Q^{-1}T^{-2}$]

4) Electrostatic field intensity E

$E = {}^{dV}/_{dx}$

The electrostatic field intensity also termed field strength **E** is $^{\text{force}}/_{\text{unit charge}}$ = and measured in $^{\textit{volts}}/_{\textit{metre}}$ or $^{\textit{newtons}}/_{\textit{coulomb}}$ [$M\ L\ Q^{-1}\ T^{-2}$]. These dimensions are the same as flux density

We can now derive the electric flux Φ_E, a vector quantity, by multiplying **E** by the unit area. ie $\int d\Phi_E = \int EdA$ so $\Phi_E = EA$ where A=unit area, with dimensions $[\, M\, L^3\, Q^{-1}\, T^{-2}\,]$

5a) Capacitance C

i = C $^{dv}/_{dt}$

Capacitance is the ability to store energy as an electric field. The units are $^{coulombs}/_{volt}$ and measured in *farads* $[\, Q^2\, T^2\, M^{-1}\, L^{-2}\,]$.

5b) Elastance $\delta = {}^{V}/_{Q}$

The inverse of capacitance is termed **Elastance** and measured in darafs $[\, M\, L^2\, Q^{-2}\, T^{-2}\,]$

6a) Permittivity ε

The ratio $^{charge\ density}/_{field\ strength}$ $^{D}/_{E}$ is termed permittivity ε and measured in $^{farads}/_{metre}$ $[\, Q^2\, T^2\, M^{-1}\, L^{-3}\,]$.

Rearranging we have **D** = ε **E** so ε is the constant of proportionality relating the two vectors **D** and **E**. It is the measure of the electric polarisation of the material.

In the earlier cgs-esu electric field strength and flux density have the same dimensions to give permittivity dimensionless. This requires charge to take the dimensions $\sqrt{(M\, L^3 T^{-2})}$.

6b) Elasticity τ

The inverse of permittivity is termed Elasticity τ measured in $^{metres}/_{farad}$ $[\, M\, L^3\, Q^{-2}\, T^{-2}\,]$. but does not have a specific name.

Magnetostatics

We now derive the corresponding terms in magnetostatics

1) Magnetic flux Φ_M

The magnetic flux Φ_M is measured in webers. The units can be derived from the definition that a change in flux of one weber per second will induce an electromotive force of one volt.

Hence the units of Φ_M are *volt-sec* $[\, M\, L^2\, Q^{-1}\, T^{-1}\,]$.

There is an alternative unit the ampere-meter where 1 Weber = μ_0 A-m

2) Flux Density B

Flux density **B** has units $^{webers}/_{metre^2}$ and measured in *teslas* $[\, M\, Q^{-1} T^{-1}\,]$.

3) Magnetomotive Force (mmf) F

F is the magnetomotive force mmf defined as the work done in moving a unit magnetic pole once around the magnetic field. It is a magnetic pressure that tends to set up magnetic flux in a magnetic field.

F is this measured in $^{joules}/_{weber}$ The dimensions all conveniently cancel out leaving [QT^{-1}]. This is termed *ampere-turns*. Note this implies an additional dimensionless constant and is not equivalent to amperes alone.

4) Magnetostatic field strength H

The magnetostatic field strength also called intensity **H** = $^{dF}/_{dx}$.

H = $^{dF}/_{dx}$ gives the dimensions of **H** as $^{AT}/_{metre}$ or $^{newtons}/_{weber}$ [$Q\,T^{-1}\,L^{-1}$]

5a) Inductance L where $v(t) = -L\,^{di}/_{dt}$

Inductance is the ability to store energy as an magnetic field. The units are $^{magnetic\ flux}/_{magnetic\ potential}$ or $^{webers}/_{ampere\text{-}turn}$ [$M\,L^2\,Q^{-2}$] and measured in *henrys*.

5b) Reluctance S

The inverse of inductance is termed and measured in sturgeons

[$Q^2\,M^{-1}\,L^{-2}$]

6a) Permeability μ

The ratio $^{flux\ density}/_{field\ strength}$ $^{\mathbf{B}}/_{\mathbf{H}}$ is termed permeability μ and measured in $^{henrys}/_{metre}$ [$M\,L\,Q^{-2}$]. It can also be expressed as $^{newtons}/_{ampere^2}$ and $^{volt\text{-}sec}/_{ampere\text{-}meter}$.

Rearranging we have **B** = μ **H** so μ is the constant of proportionality relating the two vectors **B** and **H**. It is the measure of the magnetic polarisation of the material.

In the earlier cgs-emu system magnetic field strength and flux density have the same dimensions to give permeability as dimensionless. Setting permeability as dimensionless requires charge Q to take the dimensions $\sqrt{(ML)}$

6b) Reluctivity γ

The inverse of permeability is reluctivity measured in $^{ampere\text{-}turns}/_{meter}$

[$Q^2\,M^{-1}\,L^{-1}$]

Dimensional Relationships

Summarising

Electric Flux Φ_E	**Magnetic flux Φ_M**
volts-metre.	*volt-sec (webers)*
[$ML^3Q^{-1}T^{-2}$]	[$M\,L^2Q^{-1}\,T^{-1}$]
Charge density D	**Flux Density B**
$^{coulombs}/_{metre^2}$	$^{webers}/_{metre^2}$ *(teslas)*
[$Q\,L^{-2}$]	[$M\,Q^{-1}T^{-1}$]
emf v	**mmf F**
$^{joules}/_{coulomb}$ *(volts)*	$^{joules}/_{weber}$ *(amp-turns)*
[$ML^2Q^{-1}T^{-2}$]	[QT^{-1}]
field strength E	**field strength H**
$^{newtons}/_{coulomb}$	$^{newtons}/_{weber}$
[$M\,L\,Q^{-1}\,T^{-2}$]	[$Q\,T^{-1}\,L^{-1}$]
Capacitance C	**Inductance L**
$^{coulombs}/_{volt}$ *(farads)*	$^{webers}/_{ampere\text{-}turn}$ *(henrys)*

$[\,Q^2\,T^2\,M^{-1}\,L^{-2}\,]$ $[\,M\,L^2\,Q^{-2}\,]$

Elastance δ **Reluctance** S

$^{voltsc}/_{coloumb}$ *(darafs)* $^{A\text{-}T}/_{weber}$ *(sturgeons)*

$[\,M\,L^2\,Q^{-2}\,T^{-2}\,]$ $[\,Q^2\,M^{-1}\,L^{-2}\,]$

Permittivity ε **Permeability** μ

$^{\mathbf{D}}/_{\mathbf{E}}$ $^{farads}/_{metre}$ $^{\mathbf{B}}/_{\mathbf{H}}$ $^{henrys}/_{metre}$

$[\,Q^2\,T^2\,M^{-1}\,L^{-3}\,]$ $[\,M\,L\,Q^{-2}\,]$

The dimensional relationship $^{\text{electric flux}}/_{\text{magnetic flux}}$ is $[\,ML^3\,Q^{-1}T^{-2}\,]\,[\,QT\,M^{-1}L^{-2}\,]$ which reduces to the dimensions of velocity $[\,L\,T^{-1}\,]$ and the first hint that light might be an electromagnetic phenomena.

In the electrics we have

charge = current × time and

magnetic flux = emf × time

Therefore if we replace the dimension $[\,I\,]$ with the dimension $[\,M\,L^2\,Q^{-1}T^{-2}\,]$ that gives the corresponding magnetostatic dimensions

Maxwell's 1st Law

The Law of Electric Flux and is a reformulation of Gauss's Law.

$\nabla \bullet \mathbf{D} = \rho_e$ or $\nabla \bullet \mathbf{E} = {}^{\rho}/_{\varepsilon}$

where ρ_e is the volume charge density

$[\,L^{-1}\,]\,[\,Q\,L^{-2}\,] = [\,Q\,L^{-3}\,]$

reducing to $[\,Q\,L^{-3}\,]$ either side or

$[L^{-1}][MLQ^{-1}T^{-2}] = [Q\,L^{-3}][ML^3Q^{-2}\,T^{-2}]$

reducing to $[\,M\,Q^{-1}\,T^{-2}\,]$ either side

It is a static law. It does not change with time. It states that the electric flux through a closed surface area is equal to the total charge enclosed. Derived from Gauss's Law it implies that isolated electric charges exist and that like charges repel one another while unlike charges attract. The flux lines must begin and end at a charged particle.

Maxwell's 2nd Law

The Law of Magnetic Flux is a restatement of Gauss's Law for Magnetism.

$\nabla \bullet \mathbf{B} = 0$ or $\nabla \bullet \mu\mathbf{H} = 0$

$[\,L^{-1}\,]\,[\,M\,Q^{-1}\,T^{-1}\,]$ or

$[\,L^{-1}\,]\,[\,M\,L\,Q^{-2}]\,[\,Q\,T^{-1}\,L^{-1}\,]$

reducing to $[\,M\,Q^{-1}\,T^{-1}\,L^{-1}\,]$.

This might be viewed as $^{webers}/_{metre^3}$

It is also static law and states there are no magnetic monopoles.

Maxwell's 3rd Law

The Law of Magnetic Induction is a derivation of Faraday's Law of electromagnetic induction.

$\nabla \times \mathbf{E} = -{}^{\partial \mathbf{B}}/_{\partial t}$ or $\nabla \times \mathbf{D} = \varepsilon\mu\ {}^{-\partial \mathbf{H}}/_{\partial t}$

$[L^{-1}]\ [\ MLQ^{-1}T^{-2}\ ] = [\ MQ^{-1}T^{-1}\ ]\ [T^{-1}]$

reducing to $[\ M\ Q^{-1}\ T^{-2}\ ]$ either side or

$[L^{-1}][\ Q\,L^{-2}\ ] = [T^2\ L^{-2}]\ [Q\ T^{-1}L^{-1}]\ [T^{-1}]$

where $\varepsilon\mu$ is dimensionally $[T^2\ L^{-2}]$ which is as significant as it looks and reduces to $[\ Q\,L^{-3}\ ]$ on either side.

So dimensionally these two expressions for Maxwell's 3rd Law match the two expressions for Maxwell's 1st Law.

The law states that whenever magnetic flux linked with a circuit changes then an induced electromotive force (emf) is set up in the circuit. This induced emf lasts so long as the change in magnetic flux continues. The magnitude of induced emf is equal to the rate of change of magnetic flux linked with the circuit.

Maxwell's 4th Law

is a development of Ampere's Law for non-static fields.

$\nabla \times \mathbf{H} = \mathbf{J}_c + {}^{\partial \mathbf{D}}/_{\partial t}$

$\mathbf{J}_c$ is the current density and the second term is the displacement current.

$[L^{-1}]\ [QT^{-1}L^{-1}] = [QT^{-1}L^{-2}]+[Q\ L^{-2}[T^{-1}]$

reducing to $[QT^{-1}L^{-2}]$ either side

Substituting **B** for **H** and **E** for **D** gives

$\nabla \times \mathbf{B} = \mu_0\,\mathbf{J}_c + \mu_0\,\varepsilon_0\ {}^{\partial \mathbf{E}}/_{\partial t}$

$[L^{-1}]\ [\ MQ^{-1}T^{-1}\ ] = [MLQ^{-2}][QT^{-1}L^{-2}]\ +$

$[MLQ^{-2}][Q^2T^2M^{-1}L^{-3}][MLQ^{-1}T^{-2}][T^{-1}]$

each reducing to $[\ M\ Q^{-1}\ T^{-1}\ L^{-1}\ ]$

the same dimensions as the 2nd law which we termed $^{\text{webers}}/_{\text{metre}^3}$.

As we later discover $c = \{\sqrt{(\varepsilon_0\mu_0)}\}^{-1}$

$\nabla \times \mathbf{B} = \mu_0\,\mathbf{J}_c + {}^{1}/_{c^2}\ {}^{\partial \mathbf{E}}/_{\partial t}$

Amperes Law needed adjustment for the following reason.

Nominally $\nabla \times \mathbf{H} = \mathbf{J}$

Hence $\nabla\bullet(\ \nabla\times\ \mathbf{H}\ ) = \nabla\bullet\mathbf{J}$

But this requires $\nabla\bullet\mathbf{J}$ always to be zero which is not true when we introduce a capacitor into the circuit. We therefore need to introduce an additional term

${}^{\partial \mathbf{D}}/_{\partial t} = \mathbf{J}_d$

Maxwell deduced this by pure thought and not by experimentation.

Asymmetry of Maxwell's Equations

Maxwell's 1[st] Law is

$\nabla \bullet \mathbf{D} = \rho_e$ or $\nabla \bullet \mathbf{E} = {}^{\rho}/_{\varepsilon}$

but the 2[nd] Law is $\nabla \bullet \mathbf{B} = 0$

Otherwise the equation might be

$\nabla \bullet \mathbf{B} = \mu_0 \rho_m$ or $\nabla \bullet \mathbf{H} = \rho_m$

The volume density of magnetic charge ρ_m analogous to the volume density of electric charge is ρ_e .

Maxwell's 4[th] Law is

$\nabla \times \mathbf{H} = \mathbf{J_c} + {}^{\partial \mathbf{D}}/_{\partial t}$ or

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J_c} + \mu_0 \varepsilon_0 {}^{\partial \mathbf{E}}/_{\partial t}$$

but the 3[rd] Law is

$\nabla \times \mathbf{E} = -{}^{\partial \mathbf{B}}/_{\partial t}$ or $\nabla \times \mathbf{D} = \varepsilon\, {}^{-\partial \mathbf{B}}/_{\partial t}$

Otherwise the equation might be

$\nabla \times \mathbf{E} = -\mathbf{J_m} - {}^{\partial \mathbf{B}}/_{\partial t}$ or

$$\nabla \times \mathbf{D} = -\varepsilon \mathbf{J_m} - \varepsilon {}^{\partial \mathbf{B}}/_{\partial t}$$

where $\mathbf{J_m}$ might be the magnetic current $\mathbf{J_m} = \sigma_m \mathbf{H}$ where σ_m is the magnetic conductivity.

However no magnetic monopoles are known to exist so in practical engineering problems

$\sigma_m \mathbf{H} = 0$ and hence $\mathbf{J_m} = 0$

Summary

$\nabla \bullet \mathbf{D} = \rho_e$ (pronounced div D)

$\nabla \bullet \mathbf{B} = 0$ (pronounced div B)

$\nabla \times \mathbf{E} = -{}^{\partial \mathbf{B}}/_{\partial t}$ (pronounced curl E)

$\nabla \times \mathbf{H} = \mathbf{J_c} + {}^{\partial \mathbf{D}}/_{\partial t}$ (pronounced curl H)

plus $\varepsilon = {}^{\mathbf{D}}/_{\mathbf{E}}$ and $\mu = {}^{\mathbf{B}}/_{\mathbf{H}}$

These equations tell us a time varying electric flux density **D** gives rise to a magnetic field **H** that circles the **D** field. The **H** field gives rise to an **E** field which produces the **D** field and so on. This cycle propagates an e-m wave that must move at constant velocity in three dimensional space and is independent of wavelength. The velocity must relate to the permittivity and permeability of the medium.

Wave Velocity in Free Space

Now it is immediate apparent that the ratio of Φ_E and Φ_M and consequently flux densities **D** and **B** give dimensions of velocity. In addition the dimensions of $\varepsilon_0\mu_0$ are [$T^2 L^{-2}$]. Now it is reasonable to assume the velocity of light c would depend only upon the permittivity and the permeability of free space ie $c = f(\varepsilon_0\mu_0)$. Hence by dimensional analysis alone we derive $c = \{\sqrt{(\varepsilon_0\mu_0)}\}^{-1}$. Maxwell formally deduced this demonstrating light was an electromagnetic phenomena.

Only a universe of 3 dimensions allows propagation of e-m waves at constant velocity irrespective of wavelength. A universe of say 4 or 5 dimensions would be a chaotic affair.

References

The full development of Maxwell's Equations is given in "The Introduction to Electromagnetic Fields and Waves" by Dale Corson and Paul Lorrain published in 1962. This book was given to me by Peter Hillman when having a clear-out. He immediately went up in my estimation.

A shorter development leading on to much more detailed applications is given in "Hyper and Ultrahigh Frequency Engineering" by Robert L Sarbacher and William A Edson both from the Illinois Institute of Technology, published in 1943. The book pays closer attention to the issues of dimensional analysis, exploring alternative systems but concluding the rationalised practical system due to Giorgi is the way forward ie the SI system of today.

This book belonged to by Uncle Len and retrieved after he died in. My dad described the family household where brother Len was building a crystal radio when he was around 10 in 1925 and scratching away on a cat's whisker to find a suitable hot spot.

He went on to be a radio operator in the back of Wellington Bombers in the WWII – a terrifying experience from which he never really recovered.

This and many other publications can be found at

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