Mr G's Little Book on

Logarithms

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Contents

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Introduction

Prior to the universal availability of the pocket calculator, students on reaching secondary school were handed their own copy of “Log Tables”. They rapidly learnt a new way to multiply – look up the “log” of each number, add the logs together and then search through that same table for the sum and read off the product. When arriving in the sixth form – if making it that far – they were then introduced to more sophisticated uses for “logs” both in pure and applied maths. The greatest unsolved problem in all Mathematics is rooted in logs.

Today’s student doesn’t receive that initial grounding. They are introduced to logs “cold” in the sixth form without that familiarity. They remain quirky and unfriendly and little understood.

This booklet attempts to bridge some of that gap.

1.0 First Principles

Given $a = b^n$ ①

we define $\log_b a = n$ ②

and say “log to the base b of a equals n”

Put another way we say

$$\text{lob}_b\ b^n = n \quad ③$$

So a logarithm in base b reverse the effect of a base b power.

Putting n = 1 we get

$$\text{lob}_b\ b = 1 \quad ④$$

If the subscript is not specified

$$\log a = \log_{10} a$$

The antilog of n is b^n.

The author recommends avoiding this term whenever possible.

Substituting ② into ① we get

$$b^{\log a} = a \quad ⑤$$

as long as log is base b.

The 12 Rules of Logarithms

Product Rule

Suppose we want to multiply two number M and N where $M \times N = n$.

Now we can chose x and y such that

$M = b^x$ and hence $x = \log_b M$

$N = b^y$ and hence $y = \log_b N$

for any value of b we have

$$\log_b (b^x \times b^y) = \log b^{x+y}$$

From ③ $\log_b b^{x+y} = x + y$

So we have

$$\log_b (M\times N) = \log_b M + \log_b N$$

Note that the choice of b is arbitrary

Set b = 10 for the key relationship

$$\log (M\times N) = \log M + \log N \quad \text{(i)}$$

By extension it can easily be demonstrated that

$\log (M \times N \times P \ldots) =$

$\log M + \log N + \log P \ldots \quad (i_{ext})$

Quotient Rule

As $b^x \div b^y = b^{x-y}$

then by the same reasoning

$\log (M \div N) = \log M - \log N \quad$ (ii)

Power Rule

Again setting $M = b^x$ so $x = \log_b M$

$\log_b M^k = \log_b b^{x\,k}$

$= x\, p$

$\log_b M^k = k \log_b M$

Again the base is arbitrary so we have

$\log M^k = k \log M \quad$ (iii)

So the power rule gives us the option to pull across an index p into a linear relationship with the log and gives us the "power" to solve an exponential equation.

Let $\quad 14 = 3^x$

$\log 14 = \log 3^x$

We can always take logs of either side of an equation and retain the equality.

$\log 14 = x \log 3$

$x = \log 14 / \log 3$

Another way

In ((i_{ext}) let M = N = P = … for n terms

$\log (M \times M \times M \ldots$ for n terms $) =$

$\log M + \log M + \log M \ldots$for n terms

$\log (M^n) = n \log M$

Inverse Rule

Setting $k = -1$ we have

$\log M^{-1} = -\log M$

$\log (1 / M) = -\log M \quad$ (iv)

Log Product Rule

Let $\quad b^n = a$

Taking logs of both sides to base c

$\log_c b^n = \log_c a$

$n \log_c b = \log_c a$

but we know from ② $n = \log_b a$ so

$\log_b a \log_c b = \log_c a$

Rearranging $\log_c b \times \log_b a = \log_c a$ (v)

This can be extended to any number of terms

Base Change Rule

The log product rule gives us a method to find the log to any base. Suppose we want to find the log of a to some unknown base b. Rearranging (v) gives

$\log_b a = \log_c a / \log_c b$

Setting c = 10 we have

$\log_b a = \log a / \log b \quad$ (vi)

Power Base Rule

let $\log_{b^a} c = \log_c a / \log_c b^a$

$= \log_c a / a \log_c b$

$= (1/a) \log_c a / \log_c b$

$= (1/a) \log_b c \quad$ (xvii)

In words if we know the log of a number say to base 5 then half of this value would be that number to base 25.

$\log_5 2 = 0.4307\ldots$

$\log_{25} 2 = 0.2152\ldots$

Power Base Inverse Rule

Setting a = – 1

$\log_{1/b} (1/c) = \log_b c$ (xviii)

Proportionality Rule

Recall $\log_b a = \log_c a / \log_c b$ (vi)

Setting $k = 1 / \log_c b$

$\log_b a = k \log_c a$

$\log_b a \propto \log_c a$ (xix)

Reciprocal Rule

Recalling (vi) $\log_b a = \log_c a / \log_c b$

Setting c = a

$\log_b a = \log_a a / \log_a b$

From ④ $\log a_a = 1$

$\log_b a = 1 / \log_a b$ (xx)

Combination Rule

Suppose we are faced with $\log_b a + k$

From ③ $\text{lob}_b b^k = k$

$\log_b a + k = \log_b a + \text{lob}_b b^k$

$= \log_b (a \times b^k)$ (xxi)

In words we can absorb any constant – which when we come to study calculus may be arbitrary - into a single expression.

Exponential Rule

Before we study the exponential rule we need to establish the nature of the number e.

The Nature of e

Using a graphical calculator plot the exponential curve 2^x. Using 2nd function Calc find the gradient dy/dx at x = 0

gradient = 0.69…

Repeat for exponential curve 3^x.

gradient = 1.098…

By the intermediate value theorem there must be some value of x such that the $\text{gradient}_{x=0} = 1$

We experimentation we might determine that x > 2.18 ….

e is the value where $\text{gradient}_{x=e} = 1$

e is the base rate of growth. Money invested at 3% compounded 6 monthly is a better investment than money invested at 6% compounded annually. Money inverted at 0.5% compounded monthly is better still. Imagine money that compounded every second even though at a miniscule amount.

(1.03)2 = 1.0609

(1.005)12 = 1.0616

e is the limit of (1 + 1/n)n

For n = 100

$(1 + {}^{1}/_{100})^{100} = 2.705\ldots$

For n = 1000

$(1 + {}^{1}/_{1000})^{1000} = 2.717\ldots$

For n = 1000000

$(1 + {}^{1}/_{1000000})^{1000000} = 2.7183\ldots$

It can be demonstrated rigorously that

$e = {}_{n=0}\Sigma^{\infty}\ {}^{1}/_{n!} = 1 + 1/1 + ½! + {}^{1}/_{3!}\ \ldots$

$= 2.718281828459\ldots$

Natural Logs

Although we have established that the proportionately rule states that we can move between bases simplify by introducing a constant of proportionality it will later become clear that this constant can usually be avoided if we first take the base as the value e. So by definition $\log_e x = \ln x$ and we describe as the “natural log”.

The Exponential Rule

Recalling ⑤ $a = b^{\log a}$

Setting the base as b = e

$$a = e^{\ln a}$$

Raising each side to the power x

$$a^x = e^{x \ln a}$$

Taking logs of both sides

$\ln (a^x) = \ln (e^{x \ln a})$

$x \ln a \quad = x \ln a \ln e$

so we just get $x \ln a = x \ln a$