Mr G's Little Book on

The Relativistic Energy-Momentum Equation

$$E^2 = (m_0 c^2)^2 + (pc)^2$$

Introduction

While even in common culture the relationship $E = mc^2$ *is well known, few know the full extent of its simplicity in updating Newtonian mechanics. It is often stated that Einstein's equations "when simplified" reduce to the more familiar Newtonian equations. A smaller number may appreciate that the key concept in the Special Theory of Relativity is the "inertial frame". Observers in different inertial frames may draw different conclusions as to the properties of an observed moving body – its velocity, its length and the rate of ticking of any clock moving with it.*

Yet the equation $E = mc^2$ *makes no mention of relative velocity. It's just a stark relationship between mass and energy. Given that by dimensional analysis, the dimensions of each side have to balance, if there is any relationship it couldn't really be anything else other than the introduction of a constant.*

In Newtonian mechanics the kinetic energy of a moving body is $\frac{1}{2}mv^2$ *- the greater the velocity the greater the kinetic energy. Like the mass-energy equation it's a velocity that gets squared. So where's the velocity in* $E = mc^2$ *?*

The answer - which I never appreciated until reading a book on cosmology which introduced the equation $E^2 = (m_0c^2)^2 + (pc)^2$ *without any explanation – is that the acquisition of kinetic energy is built into the expression* $E = mc^2$*. This is because the* "m" *is actually the relativistic mass which increases with velocity via the relationship* $m = m_0 / \sqrt{(1 - v^2/c^2)}$*. It is that increase in relativistic mass that automatically incorporates the increase in kinetic energy.*

So the total energy of a moving body – other than its potential energy – is given by $E = mc^2$ *and is true at any velocity. How neat is that?*

Relativistic Mass – Energy

At velocity v the relativistic mass is given by

$m = m_0 / \sqrt{(1 - v^2/c^2)}$

where m_0 is the rest mass..

To simplify further expressions we set

$\gamma = 1 / \sqrt{(1 - v^2/c^2)}$ or $(1 - v^2/c^2)^{-1/2}$

So $m = \gamma\, m_0$

Now the total energy is given by

$$E = mc^2$$

$$= \gamma\, m_0\, c^2$$

Relativistic Momentum

$$p = m\, v$$

$$= \gamma\, m_0\, v$$

Energy Momentum Equation

The task is to isolate "v" in the relativistic momentum equation and substitute that expression into the relativistic mass-energy equation.

$$p = m\, v$$

$$p^2 = (m_0 v)^2 / (1 - v^2/c^2)$$

By rearrangement we determine

$$v^2 = p^2 c^2 / (m_o^2 c^2 + p^2)$$

Now we square the relativistic mass-energy equation

E

$$= (m_0^2 c)^2 / \{ 1 - p^2c^2/(m_0^2c^2 + p^2c^2) \}$$

which by suitable rearrangement gives

$$E^2 = (m_0 c^2)^2 + (p\, c)^2$$

This relationship between the total energy of a particle in terms of momentum and rest mass was first derived by Paul Dirac in 1928.

Special Conditions

Zero Rest Mass

The photon has zero rest mass so the energy momentum equation reduces to the classic relationship between radiant energy and radiant momentum established in the 19th century.

$$E = p\, c$$

Velocity v << c

We rewrite the energy-momentum equation as

$$E = \sqrt{\{ (m_0 c^2)^2 + (p\, c)^2 \}}$$

$$= \sqrt{\{ (m_0 c^2)^2 + (p\, c)^2 \}}$$

$$= m_0 c^2 (1 + p^2 / m_0^2 c^2)^{1/2}$$

Now if v << c the relativistic mass increase is negligible and the term

$p^2 / m_0^2 c^2 << 1$

So we expand the brackets by the binomial theorem first two terms only.

$$E \approx m_0 c^2 + \tfrac{1}{2} p^2 / m_0^2 c^2$$

$$\approx m_0 c^2 + \tfrac{1}{2} m_0 v^2$$

"Ignoring" the rest mass-energy term we have the Newtonian kinetic energy

$E = \tfrac{1}{2} m v^2$ *(m the Newtonian mass)*

Paul Adrien Maurice Dirac

Paul Dirac was born in Bishopston, Bristol on 8 August 1902. He was educated local and obtained an Engineering degree at Bristol University then sitting a second degree in Mathematics before moving to Cambridge University.

He was an English theoretical physicist who is regarded as one of the most significant physicists of the 20th century. He was the Lucasian Professor of Mathematics at the University of Cambridge.

Dirac made fundamental contributions to the early development of both quantum mechanics and quantum electrodynamics – the reconciliation with special relativity. Among other discoveries, he formulated the Dirac equation which describes the behaviour of fermions and predicted the existence of antimatter. Dirac shared the 1933 Nobel Prize in Physics with Erwin Schrödinger "for the discovery of new productive forms of atomic theory".

Dirac was regarded by his friends and colleagues as unusual in character. In a 1926 letter to Paul Ehrenfest, Albert Einstein wrote of a Dirac paper, "I am toiling over Dirac. This balancing on the dizzying path between genius and madness is awful."

During questions after a lecture, a student said “Please Professor, I don’t understand the equation in the top left”. There was a long silence and eventually the moderator asked Dirac “will you be answering the question?” Dirac replied “It wasn’t a question - it was a statement”.

Dirac died 20 October 1984

The author of this paper has the publication “Lectures on Quantum Mechanics” but can’t get passed the second equation

$^{d}/_{dt}\,(\partial L/\partial \dot{q}_n) = \partial L/\partial q_n$

which requires a thorough background in partial differential equations, Lagrange’s least action principle and Hamiltonians even to get started.